FINITE ELEMENT MODELLING OF ELASTO-PLASTIC AND LARGE DEFORMATION PROBLEMS

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CERTIFICATE

This is to certify that the thesis entitled, "**FINITE ELEMENT MODELLING OF ELASTO-PLASTIC AND LARGE DEFORMATION PROBLEMS**" is a record of the work carried out by N. Sreenivasulu Reddy under our supervision and has not been submitted elsewhere for award of a degree.

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ABSTRACT

An elasto-plastic finite element code based on small deformation theory is developed for plane stress, plane strain and axisymmetric problems. In the plastic region *J_2-incremental flow theory* is used for the materials obeying *Huber-von Mises yield criterion* to model the stress-strain behaviour. Materials having elastic-perfectly plastic and elastic-linearly work hardening characteristics in uni-axial tension are considered. Code is validated considering a V-notch stress concentration problem. Metal extrusion problems have been solved to illustrate the applicability of the code. The large deformations incurred in the extrusion process is modelled using *Updated Lagrangian formulation*. Plastic work, plastic zone spread and deformed configuration are obtained for conical die extrusion. Die surface friction is also considered in modelling.

LIST OF SYMBOLS

$d\varepsilon_{ij}$	Incremental total strain tensor
$d\varepsilon^{e}_{ij}$	Incremental elastic strain tensor
$d\varepsilon^{p}_{ij}$	Incremental plastic strain tensor
$d\varepsilon'_{ij}$	Deviatoric strain tensor
$d\sigma_{ij}$	Incremental stress tensor
$d\sigma'_{ij}$	Deviatoric stress tensor
ν	Poisson's Ratio
G	Shear Modulus
E	Young Modulus
K	Bulk Modulus
δ_{ij}	Kronecker's Delta
K	Yield stress in pure shear
J_2	Second invariant of stress deviatoric tensor
$\bar{\sigma}$	Equivalent stress
$d\bar{\varepsilon}^{p}$	Equivalent plastic strain increment
$\sigma_{x}, \sigma_{y}, \sigma_{z}$	Normal stresses
$\sigma'_{x}, \sigma'_{y}, \sigma'_{z}$	Deviatoric stresses
$\tau_{xy}, \tau_{yz}, \tau_{zx}$	Shear stresses
H'	Slop of equivalent stress Vs equivalent plast strain curve.
P_c	Collapse load.

Y	Yield stress in tension
W^t	Total Work
W^e	Elastic Work
W^p	Plastic Work
$d\lambda$	Proportionality constant
$g(\sigma_{ij})$	Plastic potential function
$f(\sigma_{ij})$	Function of yield surface
$[D^{ep}]$	Elastic-plastic stress-strain matrix
$[D^e]$	Elastic stress-strain matrix
$[K]$	Total stiffness matrix
$[K^e]$	Elastic stiffness matrix of an element
$[K^{ep}]$	Elastic-Plastic stiffness matrix of an element
$[C^e]$	Elastic compliance matrix
$\{F\}$	Equilibrium force vector
$\{q\}$	Body force vector
$[N]$	Shape function matrix
$\{R\}$	Total load vector applied externally
$\{\Delta R\}$	Incremental applied load vector
$\{ \}$	Vector
$[ ]$	Matrix

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CHAPTER 1

INTRODUCTION

1.1 MOTIVATION

In general most of the structural materials are Elastic-Plastic in nature. The complete behaviour of the materials will be well understood if we know the behaviour of the materials in the plastic range. The non_linear stress-strain relations, the loading path dependence and the large deformations in the plastic range make the analysis tedious. Over the years the Finite Element Method has been successfully employed in analysing the structural behaviour in elastic as well as elasto-plastic range.

Moreover in metal forming processes like extrusion, forming and drawing etc., the given material is deformed in the complete plastic range so as to get the desired shape. *Visco-plastic* models are commonly employed in understanding these material processes. In these processes there are two distinct regions (a) elastic region (b) elastic-plastic region. In the second region where dominantly plastic behaviour occurs, visco-plastic models are employed. It is necessary to define the boundary of this plastic region when visco-plastic material is employed in order to get the good estimate of power required, finished product shape and forming pressure.

Keeping the above points in view the present thesis is mainly devoted to the elasto-plastic finite element implementation and extending towards the application of metal extrusion problems.

1.2 Aim And Scope Of The Present Work

An incremental Finite Element Analysis has been developed for the elasto-plastic problems. The plastic stress-strain relations in the plastic range are modelled using *Prandtl-Reuss relation*. The following methods are employed in solving the final discretised continuum equations viz. *Newton-Raphson method, Modified Newton-Raphson method* and *Yamada's* [1] *approach*. In the later approach an increment load sufficient enough to yield one element at a time is applied and the loads are incremented accordingly. Constant stress/strain triangular (3 - nodal) elements have been employed. Materials having characteristics in uni-axial tension as elastic-perfectly plastic and elastic-linearly work hardening are considered.

1.3 Outline Of The Thesis

In the present chapter motivation, aim and scope and out line of the thesis are presented. The review of the literature for the present work is given in chapter 2. In chapter 3 the elastic and plastic stress-strain matrix, finite element formulation for *elasto-plastic analysis* and all constitutive relations for plane-strain, plane-stress and axisymmetric cases are derived. Some of the numerical methods for solving non-linear problems are presented in the same chapter. Formulation and flow sequence of the program for the extrusion problems are given in chapter 4. Results and discussion are presented in chapter 5. In chapter 6 conclusions and scope for the future work are presented.

CHAPTER 2

LITERATURE SURVEY

Marcal and King [4] suggested a stiffness method for elastic-plastic problems. They obtained the incremental stress strain relations for an elastic-plastic materials in terms of *partial stiffness coefficients*. The partial stiffness coefficients for the solid body of revolution with symmetric loading are given as follows.

With cylindrical coordinates θ, r, z and using the conventional notations for the stresses, the *von Mises yield criterion* can be written as [4]

$$(\sigma_\theta - \sigma_r)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2 = 2\,\bar{\sigma}^2 \tag{2.1}$$

Equation (2.1) can be put in an implicit differential form as

$$3\,\sigma'_\theta\, d\sigma_\theta + 3\,\sigma'_r\, d\sigma_r + 3\,\sigma'_z\, d\sigma_z + 6\,\sigma_{rz}\, d\sigma_{rz} = 2\,\bar{\sigma}\, d\bar{\sigma} \tag{2.2}$$

where

σ' denotes the deviatoric stress e.g. $\sigma'_\theta = \frac{1}{3}(\,2\sigma_\theta - \sigma_r - \sigma_z)$

$\bar{\sigma}$ is the equivalent stress

The prefix d in eq. (2.2) is used to denote the changes over the small increment of the load.

Marcal and King [4] used the following incremental stress strain relation

$$\{d\varepsilon\} = [\sigma_m]\,\{d\sigma\} \tag{2.3}$$

where $[\sigma_m]$ is the 5 X 5 symmetric matrix. $\{d\sigma\}$ and $\{d\varepsilon\}$ are column vectors of stress and strain.

Equation (2.3) can be written as

$$\{d\sigma\} = [\sigma_m]^{-1}\{d\varepsilon\} \tag{2.4}$$

$[\sigma_m]^{-1}$ is called as *partial stiffness matrix* and can be written as

$$[\sigma_m]^{-1} = \frac{\partial\sigma_i}{\partial\varepsilon_j} \quad \text{for} \quad i,j = \theta,\ r,\ z,\ rz,\ p$$

Here σ_p is used to represent $\bar{\varepsilon}^{\,p}$ and ε_p equal to zero. The coefficients $\partial\sigma_i / \partial\varepsilon_j$ are called *partial stiffness coefficients*.

In using the above partial stiffness method, complications arise when the elastic part of the structure with stresses near the yield stress becomes plastic with the next increment of the load. The region which is adjacent to the elastic-plastic interface had been called the *transition region*. For this Marcal and King [4] defined a mean stiffness of the element by weighting of the elastic and elastio-plastic stiffness in the following ratio.

$$[\sigma_m]^{mean} = m\,[\sigma_m]^{e} + (1-m)\,[\sigma_m]^{ep}$$

where the superscript ep indicates elastic-plastic and m is the strain required to yield / estimated strain for the increment of load

Later *Yamada, Yashimura and Sukurai* [1] obtained an explicit relation for $[D^{ep}]$ (i.e. elasto-plastic stress strain matrix) for von Mises materials using *Prandtl-Reuss relations*. The expression obtained takes a simple form and can be readily accommodated to the finite element analysis. At each incremental stage of calculation which traces the expansion of the elastic-plastic interface. This method is similar to that of *Marcal and King*[4], but uses the small and varying increments of load is applied in every iteration.

The procedures of calculation and flow sequence of Yamada's method are given in chapter 3. The present thesis uses the Yamada's

method for elasto-plastic analysis.

Lea and Kobayashi [2] analyse the flat punch indentation problem for elastic-plastic boundaries and for stress distributions by using *Yamada's* algorithm.

Iwata, Osakad and Fujino [3] did the Elasto-Plastic analysis of hydrostatic extrusion using the same *Yamada's* technique.

CHAPTER 3

FORMULATION OF ELASTO-PLASTIC PROBLEM

In this chapter necessary constitutive relations for materials obeying *Huber-von Mises yield criterion* and following *Prandtl-Reuss incremental flow theory*(J_2-incremental plasticity flow theory/ Von Mises flow rule) are presented. The material is assumed to be homogeneous and isotropic during the entire loading path. The behavior of the material in uniaxial tension is characterized by Elastic - perfectly plastic and Elastic -linear work-hardening nature.

Fig. 3.1a shows the uniaxial tension stress-strain relations for Elastic-perfectly plastic model and are expressed mathematically as

$$\varepsilon = \frac{\sigma}{E} \qquad \text{for} \quad \sigma \leq \sigma_y \tag{3.1}$$

$$\varepsilon = \frac{\sigma}{E} + \lambda \qquad \text{for} \quad \sigma = \sigma_y \tag{3.2}$$

where

E is young's modulus

ε is uniaxial strain

σ is uniaxial stress

σ_y is yield stress

λ is a scalar to be determined and is greater than 0.

Fig. 3.1b shows the stress strain relations for elastic-linear work hardening model and are expressed as

$$\varepsilon = \frac{\sigma}{E} \qquad \text{for} \quad \sigma \leq \sigma_y \tag{3.3}$$

$$\varepsilon = \frac{\sigma_y}{E} + \frac{1}{E_t}(\sigma-\sigma_y) \qquad \text{for} \quad \sigma > \sigma_y \tag{3.4}$$

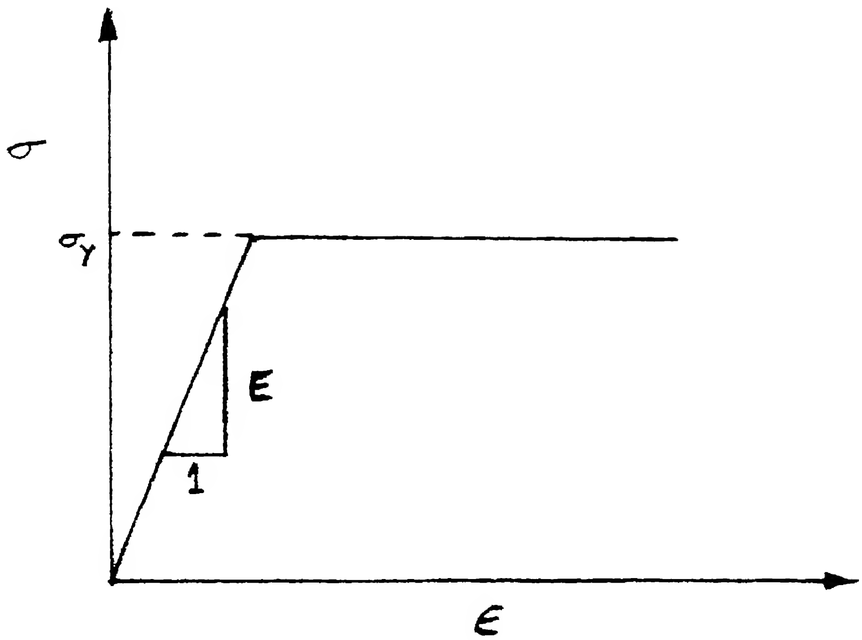


Fig. 3.1a Stess strain diagram for linearly elastic-perfectly plastic material

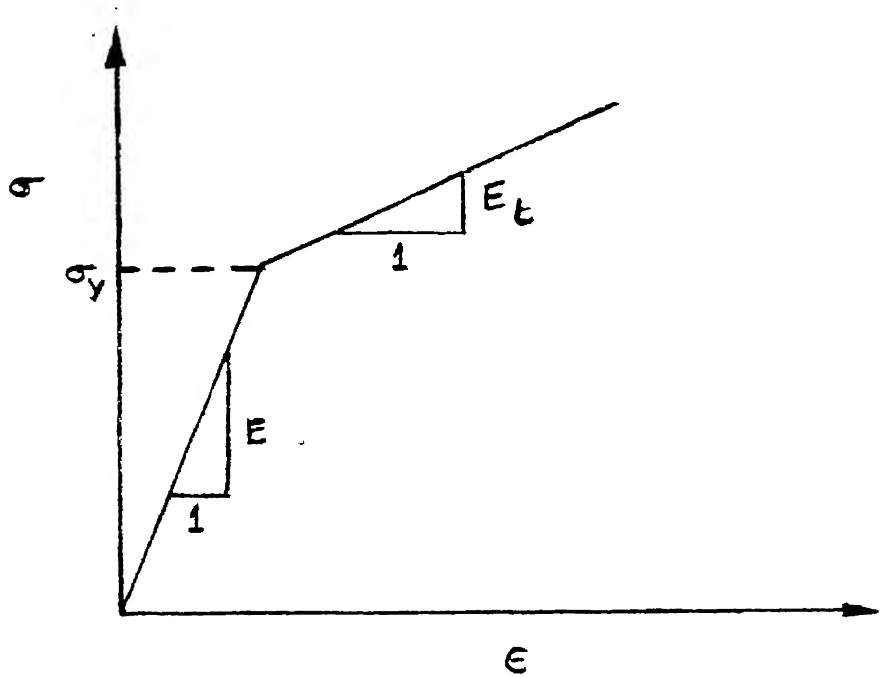


Fig. 3.1b Stress strain diagram for linearly elastic - linear work hardening material

where

E_t is tangential modulus.

The *Small strain theory* is applied for every incremental load. The loadings are monotonic and increasing in nature. The loadings are also considered to be proportional in nature. *Updated Lagrangian formulation* is used to get the deformed configuration.

In the following sections ε_{ij} represents tensorial strain and $\{\varepsilon\}$ represents engineering strain vector.

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_z}{\partial z} \\ \dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y} \\ \dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y} \\ \dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z} \end{Bmatrix}$$

where u_x, u_y and u_z are functions in x, y and z which give displacements in x, y and z directions respectively.

3.1 Elastic Stress-Strain Matrix

The total strain increments $d\varepsilon_{ij}$ are made up of increments $d\varepsilon^{e}_{ij}$ of the elastic strain components and increments $d\varepsilon^{p}_{ij}$ of the

plastic strain components.

$$d\varepsilon_{ij} = d\varepsilon^{e}_{ij} + d\varepsilon^{p}_{ij} \tag{3.5}$$

The elastic strain increments are related to the increments of stress ($d\sigma_{ij}$) by Hook's law.

$$d\varepsilon^{e}_{ij} = \frac{1+\nu}{E}\, d\sigma_{ij} - \frac{\nu}{E}\, d\sigma_{kk}\delta_{ij} \tag{3.6}$$

The equation(3.6) can be solved in terms of stress increments

$$d\sigma_{ij} = \frac{E}{(1+\nu)}\, d\varepsilon^{e}_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\, d\varepsilon_{kk}\delta_{ij} \tag{3.7}$$

Strain increments can also be written in terms of deviatoric (σ'_{ij})stress. Equation (3.6) becomes

$$d\varepsilon^{e}_{ij} = \frac{1+\nu}{E}\, d\sigma'_{ij} + \frac{(1-2\nu)}{3E}\, d\sigma_{kk}\delta_{ij} \tag{3.8}$$

Where

ν is the poisson's ratio

δ_{ij} is Kronecker's Delta $= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

The equation (3.7) can be written in the matrix form as

$$\{\sigma\} = [D^{e}]\,\{\varepsilon\} \tag{3.9}$$

Where

$[D^{e}]$ is the elastic constitutive or elastic moduli matrix

$\{\varepsilon\}$ is the elastic strain vector

$$\{\varepsilon\}^{T} = [\ \varepsilon_{x} \quad \varepsilon_{y} \quad \varepsilon_{z} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}\ ]$$

$\{\sigma\}$ is the stress vector

$$\{\sigma\}^{T} = [\ \sigma_{x} \quad \sigma_{y} \quad \sigma_{z} \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}\ ]$$

$$[D^e] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ & (1-\nu) & \nu & 0 & 0 & 0 \\ & & (1-\nu) & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & \text{Symmetric} & & & \frac{(1-2\nu)}{2} & 0 \\ & & & & & \frac{(1-2\nu)}{2} \end{bmatrix}$$

Equation (3.6) for total stress and strain can be written in the matrix form as

$$\{\varepsilon^e\} = [C^e]\{\sigma\} \tag{3.10}$$

Where

$[C^e]$ is elastic compliance matrix

$$[C^e] = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & \text{Symmetric} & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix}$$

3.2 Elasto-Plastic Stress-Strain matrix

An explicit expression of elastic-plastic stress-strain matrix $[D^{ep}]$ had been obtained by *Yamada*[1] The plastic strain increment $(d\varepsilon^{p}_{ij})$ in terms of plastic potential function $g(\sigma_{ij})$ is given by [1]

$$d\varepsilon^{p}_{ij} = d\lambda \frac{\partial g}{\partial \sigma_{ij}} \tag{3.11}$$

$d\lambda$ is a positive scalar factor of proportionality, which is non zero only when plastic deformation occurs or plastic loading occurs.

The associated flow rule is obtained by considering yield function(f) for plastic potential in equation (3.11)

$$d\varepsilon^{p}_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{3.12}$$

$$f(\sigma_{ij}) = g(\sigma_{ij}) = J_2 - K^2$$

where

K is the yielding shear stress

J_2 is second invariant of the deviatoric stress tensor

Equation (3.12) takes the simple form as

$$d\varepsilon^{p}_{ij} = d\lambda\, \sigma'_{ij} \tag{3.13}$$

The above equation is called *Prandtl-Reuss equation.*

Using equations (3.5) and (3.8) the Prandtl-reuss equation for the total deviatoric strain increment $d\varepsilon'_{ij}$ during continued loading are expressed as

$$d\varepsilon'_{ij} = \sigma'_{ij} d\lambda + \frac{d\sigma'_{ij}}{2G} \tag{3.14}$$

Where

$$d\lambda = \frac{3}{2}\,\frac{d\bar{\varepsilon}^{\,p}}{\bar{\sigma}} = \frac{3}{2}\,\frac{d\bar{\sigma}}{\bar{\sigma}\,H'} \tag{3.15}$$

The derivation for $d\lambda$ is given in the appendix -I.

Equivalent stress $\bar{\sigma}$ and plastic strain increment $d\bar{\varepsilon}^{\,p}$ are given by

$$\bar{\sigma} = \sqrt{\frac{3}{2}\,\sigma'_{ij}\sigma'_{ij}} \tag{3.16a}$$

$$\bar{\sigma}^{\,2} = \frac{3}{2}\left[(\sigma_x - \sigma_m)^2 + (\sigma_y - \sigma_m)^2 + (\sigma_z - \sigma_m)^2 + 2\tau_{xy}^2 + 2\tau_{yz}^2 + 2\tau_{zx}^2\right]$$

Where

$$\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

After Simplification

$$\bar{\sigma} = \left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x + 3\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]^{\frac{1}{2}}$$

$$d\bar{\varepsilon}^{\,p} = \sqrt{\frac{2}{3}\,d\varepsilon_{ij}^{p}\,d\varepsilon_{ij}^{p}} \tag{3.16b}$$

$H' = \dfrac{d\bar{\sigma}}{d\bar{\varepsilon}^{\,p}}$ corresponds to the slope of the curve equivalent stress $\bar{\sigma}$ Vs plastic strain $(\int d\bar{\varepsilon}^{\,p})$. $d\bar{\sigma}$ is the increment in the equivalent stress

The Von Mises yield criteria is given by

$$\sigma'_{ij}\,\sigma'_{ij} = \frac{2}{3}\,\bar{\sigma}^{\,2} \tag{3.17}$$

Taking the differentials on both the sides of the above equation

$$\sigma'_{ij}\, d\sigma'_{ij} = \frac{2}{3}\, \bar{\sigma}\, d\bar{\sigma}$$

Making use of equation (3.15) the above equation can be written as

$$\sigma'_{ij}\, d\sigma'_{ij} = \frac{4}{9}\, \bar{\sigma}^{2}\, H'\, d\lambda \qquad (3.18)$$

Eliminating $d\sigma'_{ij}$ from equation (3.14) and equation (3.18)

$$2G\, \sigma'_{ij} \left[d\varepsilon'_{ij} - \sigma'_{ij}\, d\lambda \right] = \frac{4}{9}\, \bar{\sigma}^{2}\, H'\, d\lambda$$

Rearranging the terms to solve for $d\lambda$

$$d\lambda \left[\frac{4}{9}\, \bar{\sigma}^{2}\, H' + 2G\, \sigma'_{ij}\, \sigma'_{ij} \right] = 2G\, \sigma'_{ij}\, d\varepsilon'_{ij}$$

$$d\lambda = \frac{2G\, \sigma'_{ij}\, d\varepsilon'_{ij}}{\left[\frac{4}{9}\, \bar{\sigma}^{2}\, H' + 2G\, \sigma'_{ij}\, \sigma'_{ij} \right]}$$

$$d\lambda = \frac{\sigma'_{ij}\, d\varepsilon'_{ij}}{\frac{2}{3}\, \bar{\sigma}^{2} \left[1 + \frac{H'}{3G} \right]}$$

$$d\lambda = \frac{\sigma'_{ij}\, d\varepsilon'_{ij}}{S} \qquad (3.19)$$

With

$$S = \frac{2}{3}\, \bar{\sigma}^{2} \left[1 + \frac{H'}{3G} \right] \qquad (3.20)$$

Equation (3.19) can be written as

$$d\lambda = \frac{\sigma'_{ij}\, d\varepsilon'_{ij}}{S} = \frac{\sigma'_{ij}\, d\varepsilon_{ij}}{S} \qquad (3.21)$$

$\sigma'_{ij}\, d\varepsilon'_{ij} = \sigma'_{ij}\, d\varepsilon_{ij}$ since $\sigma'_{ii} = \sigma'_{x} + \sigma'_{y} + \sigma'_{z}$ is identically zero.

Deviatoric strain tensor $d\varepsilon'_{ij}$ is defined as

$$d\varepsilon'_{ij} = d\varepsilon_{ij} - \delta_{ij} \frac{d\varepsilon_{kk}}{3} \tag{3.22}$$

Where

$$d\varepsilon_{kk} = d\varepsilon_x + d\varepsilon_y + d\varepsilon_z$$

Now substituting $d\lambda$ of equation (3.21) back into equation (3.14) and making use of equation (3.22), the deviatoric stress increment tensor can be expressed as

$$d\sigma'_{ij} = 2G\left[d\varepsilon'_{ij} - \sigma'_{ij} \frac{\sigma'_{lm}\, d\varepsilon_{lm}}{S} \right]$$

$$d\sigma'_{ij} = 2G\left[d\varepsilon_{ij} - \delta_{ij} \frac{d\varepsilon_{kk}}{3} - \frac{\sigma'_{ij}\, \sigma'_{lm}\, d\varepsilon_{lm}}{S} \right] \tag{3.23}$$

The total stress increment $d\sigma_{ij}$ is defined as

$$d\sigma_{ij} = d\sigma'_{ij} + \frac{\delta_{ij}}{3}\, d\sigma_{kk}$$

$$= d\sigma'_{ij} + \frac{E}{3(1-2\nu)} \delta_{ij}\, d\varepsilon_{ii}$$

$$= d\sigma'_{ij} + \frac{2(1+\nu)G}{3(1-2\nu)} \delta_{ij}\, d\varepsilon_{ii}$$

Finally, substituting for $d\sigma'_{ij}$ from equation (3.23) yields

$$d\sigma_{ij} = 2G\left[d\varepsilon_{ij} + \frac{\nu}{(1-2\nu)} \delta_{ij}\, d\varepsilon_{ij} - \sigma'_{ij} \frac{\sigma'_{kl}\, \sigma'_{kl}}{S} \right] \tag{3.24}$$

The equation is represented in matrix form as

$$\{d\sigma\} = [D^{ep}]\, \{d\varepsilon\} \tag{3.25}$$

The plastic stress-strain matrix $[D^{ep}]$ is symmetric and is expressed in the explicit form.

$[D^{ep}]$ is given by

$$\frac{E}{(1+\nu)}\begin{bmatrix} a-\frac{\sigma_x'^2}{S} & & & & & \\ b-\frac{\sigma_x'\sigma_y'}{S} & a-\frac{\sigma_y'^2}{S} & & \text{Symmetric} & & \\ b-\frac{\sigma_x'\sigma_z'}{S} & b-\frac{\sigma_z'\sigma_y'}{S} & a-\frac{\sigma_z'^2}{S} & & & \\ -\frac{\sigma_x'\tau_{xy}}{S} & -\frac{\sigma_y'\tau_{xy}}{S} & -\frac{\sigma_z'\tau_{xy}}{S} & \frac{1}{2}-\frac{\tau_{xy}^2}{S} & & \\ -\frac{\sigma_x'\tau_{yz}}{S} & -\frac{\sigma_y'\tau_{yz}}{S} & -\frac{\sigma_z'\tau_{yz}}{S} & -\frac{\tau_{xy}\tau_{yz}}{S} & \frac{1}{2}-\frac{\tau_{yz}^2}{S} & \\ -\frac{\sigma_x'\tau_{zx}}{S} & -\frac{\sigma_y'\tau_{zx}}{S} & -\frac{\sigma_z'\tau_{zx}}{S} & -\frac{\tau_{zx}\tau_{xy}}{S} & -\frac{\tau_{zx}\tau_{yz}}{S} & \frac{1}{2}-\frac{\tau_{zx}^2}{S} \end{bmatrix}$$

Where

$$a = \frac{(1-\nu)}{(1-2\nu)}$$

$$b = \frac{\nu}{(1-2\nu)}$$

3.3 Finite Element Formulation

The general governing equation of the finite element method for static analysis can be derived from the *principle of virtual work*,

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\,dV = \int_A F_i\delta u_i\,dA + \int_V q_i\delta u_i\,dV \tag{3.26}$$

where δu_i and $\delta\varepsilon_{ij}$ are virtual displacement and virtual strain increments, and they form a compatible set of deformations. F_i and q_i are surface traction and body forces,V and A represents volume and area of integration respectively and σ_{ij} with F_i and q_i form the equilibrium set. In a matrix form Eq.(3.26) becomes

$$\int_V \{\delta\varepsilon\}^T\{\sigma\}\, dV = \int_A \{\delta u\}^T\{F\}\, dA + \int_V \{\delta u\}^T\{q\}\, dV \tag{3.27}$$

where the vectors for displacements $\{ u \}$, strain $\{ \varepsilon \}$, and stress $\{\sigma\}$ are defined as

$$\{u\}^T = [\, u_1 \quad u_2 \quad u_3 \,], \qquad \{\delta u\}^T = [\, \delta u_1 \quad \delta u_2 \quad \delta u_3 \,] \tag{3.28}$$

$$\{\varepsilon\}^T = [\, \varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx} \,] \tag{3.29a}$$

$$\{\delta\varepsilon\}^T = [\, \delta\varepsilon_x \quad \delta\varepsilon_y \quad \delta\varepsilon_z \quad \delta\gamma_{xy} \quad \delta\gamma_{yz} \quad \delta\gamma_{zx} \,] \tag{3.29b}$$

$$\{\sigma\}^T = [\, \sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx} \,] \tag{3.30}$$

For a geometrically linear analysis or a small deformation analysis, we have,

$$\{\varepsilon\} = [B]\, \{U\} \qquad \{\delta\varepsilon\} = [B]\, \{\delta U\} \tag{3.31}$$

where $\{U\}$ is the displacement vector for nodal points that is related to the distributed displacement $\{u\}$ by,

$$\{u\} = [N]\, \{U\} \tag{3.32}$$

in which [N] is the matrix of *displacement interpolation function*, or the *shape function* , and the *strain displacement matrix* [B] is a matrix defined as

$$[B] = [L]\, [N] \tag{3.33}$$

and [L] is the *differential operator matrix* defined as

$$L = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \tag{3.34}$$

such that

$$\{\varepsilon\} = [L]\ \{u\} \tag{3.35}$$

Substituting the Eqs. (3.31) and (3.32) in Eq. (3.27), the governing equation for a small-deformation analysis is

$$\int_V [B]^T\{\sigma\}\ dV = \int_A [N]^T\{F\}\ dA + \int_V [N]^T\{q\}\ dV \tag{3.36a}$$

or

$$\int_V [B]^T\{\sigma\}\ dV = \{R\} \tag{3.36b}$$

where {R} is equivalent external force acting on the nodal points,

$$\{R\} = \int_A [N]^T\{F\}\ dA + \int_V [N]^T\{q\}\ dV \tag{3.37}$$

Therefore, using $\{\sigma\} = [D]\{\varepsilon\} = [D][B]\{U\}$ the governing equation can be written as,

$$[K]\ \{U\} = \{R\} \tag{3.38a}$$

In incremental form

$$[K]\ \{dU\} = \{dR\} \tag{3.38b}$$

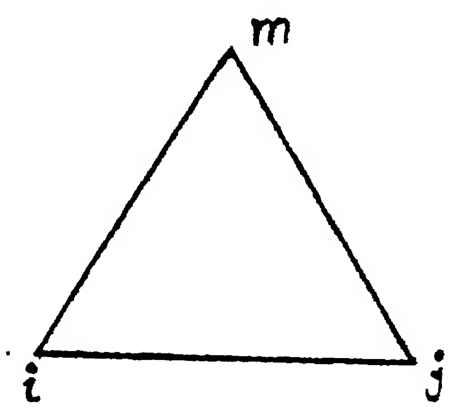
where [K] is the stiffness matrix of the structure ,

$$[K] = \int_V [B]^T[D]\ [B]\ dV \tag{3.38c}$$

in which [D] is the constitutive matrix.

For elastic case [D] is replaced by $[D^e]$ and for plastic case [D] is replaced by $[D^{ep}]$.

The whole domain is divided into triangular elements. The following figure shows one such triangular element with three nodes i, j and m.



The matrix [B] is given by

$$[B] = [B_i \quad B_j \quad B_m]$$

$$[B_i] = \frac{1}{2\,\Delta}\begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix}$$

Where

$$b_i = y_j - y_m$$

$$c_i = x_m - x_j$$

x_i, y_i are the x and y co-ordinates of the i^{th} node in the element

Δ = Area of the the element

$$= \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) \right]$$

Incremental Analysis

In this analysis equation (3.38b) has been converted into

$$[K]\{\Delta U\} = \{\Delta R\}$$

$$[K]^m\{U^{m+1} - U^m\} = \{R^{m+1} - R^m\}$$

[K] is evaluated at m^{th} increment and the problem is solved for U^{m+1}

In an incremental analysis, the total load {R} acting on a structure is added in increments step by step. At the $(m+1)^{th}$ step, the load can be expressed as

$$^{m+1}\{R\} = {}^{m}\{R\} + {}^{m+1}\{\Delta R\} \tag{3.39}$$

where the left superscript *m* has been used to indicate the

m^{th} incremental step. Assuming the solution at the m^{th} step, ${}^{m}\{U\}$, ${}^{m}\{\sigma\}$, ${}^{m}\{\varepsilon\}$, are known, and at the $(m+1)^{th}$ step, corresponding to the load increment $\{\Delta R\}$,

$$ {}^{m+1}\{\ U\ \} = {}^{m}\{\ U\ \} + \{\ \Delta U\ \} \tag{3.40} $$

$$ {}^{m+1}\{\ \sigma\ \} = {}^{m}\{\ \sigma\ \} + \{\ \Delta\sigma\ \} \tag{3.41} $$

Here, the left superscript for the increments has been dropped. Eq. (3.36) becomes

$$ {}^{m+1}\{\ F\ \} = {}^{m+1}\{\ R\ \} \tag{3.42a} $$

where ${}^{m+1}\{\ F\ \}$ is equivalent force of stress acting on the nodal points,

$$ {}^{m+1}\{F\} = \int_V [B]^T\ {}^{m+1}\{\sigma\}\ dV \tag{3.42b} $$

or

$$ \int_V [B]^T\ \{\Delta\sigma\}\ dV = {}^{m+1}\{R\} - \int_V [B]^T\ {}^{m}\{\sigma\}\ dV \tag{3.42c} $$

Eq. (3.42c), in fact, represents the equilibrium of the external forces, ${}^{m+1}\{R\}$, with the internal forces ${}^{m+1}\{F\}$. Two types of algorithms are therefore involved in solving Eq. (3.35) for the stress increment $\{\ \Delta\sigma\ \}$ and displacement increment $\{\ \Delta U\ \}$.

3.4 Special Cases

1. Plane Stress

Elastic case

In this case

$$\sigma_z = 0 \; ; \; \tau_{yz} = 0 \; ; \; \tau_{xz} = 0 \quad \text{and}$$

$$\gamma_{yz} = 0 \; ; \; \gamma_{xz} = 0 \; ; \; \varepsilon_z^e = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

The stress vector is reduced to

$$\{\sigma\} = [\, \sigma_x \;\; \sigma_y \;\; \tau_{xy} \,]^T$$

The strain vector is reduced to

$$\{\varepsilon\} = [\, \varepsilon_x \;\; \varepsilon_y \;\; \gamma_{xy} \,]^T$$

Elastic moduli matrix

$$[D^e] = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

Elastic compliance matrix

$$[C^e] = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}$$

Plastic case

$$[D^{ep}] = \frac{E}{Q} \begin{bmatrix} \sigma'^{2}_y + 2p & & \text{Symmetric} \\ -\sigma'_x \sigma'_y + 2\nu p & \sigma'^{2}_x + 2p & \\ -\frac{\sigma'_x + \nu\sigma'_y}{(1+\nu)}\tau_{xy} & -\frac{\sigma'_y + \nu\sigma'_x}{(1+\nu)}\tau_{xy} & \frac{R}{2(1+\nu)} + \frac{2H'}{9E}(1-\nu)\bar{\sigma}^2 \end{bmatrix}$$

Where

$$P = \frac{2H'}{9E}\bar{\sigma}^2 + \frac{\tau_{xy}^2}{(1+\nu)}$$

$$R = \sigma_x'^2 + 2\nu\sigma_x'\sigma_y' + \sigma_y'^2$$

$$Q = R + 2P(1-\nu^2)$$

$$\bar{\sigma} = \left[\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 \right]^{\frac{1}{2}}$$

$$\bar{\varepsilon} = \frac{2}{3}\left\{ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x\varepsilon_y - \varepsilon_z\varepsilon_y - \varepsilon_x\varepsilon_z + \frac{3}{4}\gamma_{xy} \right\}^{\frac{1}{2}}$$

2.Plane Strain

Elastic case

In this case

$\varepsilon_z = 0$; $\gamma_{yz} = 0$; $\gamma_{zx} = 0$; and

$\tau_{yz} = 0$; $\tau_{zx} = 0$; $\sigma_z = \dfrac{E\,\nu}{(1+\nu)(1-2\nu)}(\varepsilon_x + \varepsilon_y)$

The stress vector is reduced to

$$\{\sigma\} = [\ \sigma_x \quad \sigma_y \quad \tau_{xy}\ ]^T$$

The strain vector is reduced to

$$\{\varepsilon\} = [\ \varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\ ]^T$$

$$[D^{e}] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}$$

$$[C^{e}] = \frac{(1+\nu)}{2} \begin{bmatrix} (1-\nu) & -\nu & 0 \\ -\nu & (1-\nu) & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

For plastic case

$$[D^{ep}] = \frac{E}{(1+\nu)} \begin{bmatrix} \frac{(1-\nu)}{(1-2\nu)} - \frac{\sigma_x'^2}{S} & \frac{\nu}{(1-2\nu)} - \frac{\sigma_x' \sigma_y'}{S} & -\frac{\sigma_x' \tau_{xy}}{S} \\ & \frac{(1-\nu)}{(1-2\nu)} - \frac{\sigma_y'^2}{S} & -\frac{\sigma_y' \tau_{xy}}{S} \\ \text{Symmetric} & & \frac{1}{2} - \frac{\tau_{xy}^2}{S} \end{bmatrix}$$

In this case stress component in z-direction is not equal to zero and is given by

$$d\sigma_z = \frac{E}{(1+\nu)} \left[\left(\frac{\nu}{(1-2\nu)} - \frac{\sigma_x' \sigma_z'}{S} \right) d\varepsilon_x^p + \left(\frac{\nu}{(1-2\nu)} - \frac{\sigma_y \sigma_z'}{S} \right) d\varepsilon_y^p - \frac{\sigma_z' \tau_{xy}}{S} d\gamma_{xy}^p \right]$$

Equivalent stress $\bar{\sigma}$ is given by

$$\bar{\sigma} = \left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x + 3 \tau_{xy}^2 \right)^{\frac{1}{2}}$$

and $\bar{\varepsilon}$ is given by

$$\bar{\varepsilon} = \frac{2}{3}\left[\varepsilon_x^2 + \varepsilon_y^2 - \varepsilon_x\varepsilon_y + \frac{3}{4}\gamma_{xy}\right]^{\frac{1}{2}}$$

[B] matrix for this case is same as in the plane stress case

3.Axisymmetry

Elastic case

In this case

$\tau_{\theta z} = 0$; $\tau_{r\theta} = 0$; and

$\gamma_{\theta z} = 0$; $\gamma_{r\theta} = 0$;

The stress vector becomes

$$\{\sigma\} = [\ \sigma_r \quad \sigma_z \quad \sigma_\theta \quad \tau_{rz}\ ]^T$$

and the strain vector becomes

$$\{\varepsilon\} = \{\ \varepsilon_r \quad \varepsilon_z \quad \varepsilon_\theta \quad \gamma_{rz}\}^T$$

r and θ represent the radial and tangential directions respectively.

$$[D^e] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 \\ & (1-\nu) & \nu & 0 \\ & & (1-\nu) & 0 \\ \text{Symmetric} & & & \frac{(1-2\nu)}{2} \end{bmatrix}$$

$$[C^e] = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 \\ & 1 & -\nu & 0 \\ & & 1 & 0 \\ \text{Symmetric} & & & 2(1+\nu) \end{bmatrix}$$

For plastic case

$$[D^{ep}] = \frac{E}{(1+\nu)} \begin{bmatrix} \frac{(1-\nu)}{(1-2\nu)} - \frac{\sigma'^2_r}{S} & \frac{\nu}{(1-2\nu)} - \frac{\sigma'_r \sigma'_z}{S} & \frac{\nu}{(1-2\nu)} - \frac{\sigma'_r \sigma'_\theta}{S} & - \frac{\sigma'_r \tau_{rz}}{S} \\ & \frac{(1-\nu)}{(1-2\nu)} - \frac{\sigma'^2_z}{S} & \frac{\nu}{(1-2\nu)} - \frac{\sigma'_z \sigma'_\theta}{S} & - \frac{\sigma'_z \tau_{rz}}{S} \\ \text{Symmetric} & & \frac{(1-\nu)}{(1-2\nu)} - \frac{\sigma'^2_\theta}{S} & - \frac{\sigma'_\theta \tau_{rz}}{S} \\ & & & \frac{1}{2} - \frac{\tau_{rz} \tau_{rz}}{S} \end{bmatrix}$$

3.5 NUMERICAL ALGORITHMS FOR SOLVING NONLINEAR EQUATIONS

In this section ,the two methods of the Newton type that have

been widely used in finite element analysis are discussed.

Considering that the stress $\{ \sigma \}$ is a nonlinear function of displacement , $\{U\}$, Eq (3.42a) can be rewritten as

$$\Psi({}^{m+1}\{ U \}) = {}^{m+1}\{ F ({}^{m+1}\{ U \}) \} - {}^{m+1}\{ R \} \tag{3.44}$$

Eq (3.44) is a nonlinear matrix equation expressed in terms of displacements ${}^{m+1}\{ U \}$. As noted previously in Section 3.3, this equation represents an equilibrium of the external forces, ${}^{m+1}\{R\}$, with the internal forces , ${}^{m+1}\{F\}$. The iterative method used to solve Eq. (3.40) is therefore called the *equilibrium iterative method*.

3.5.1 Newton-Raphson Method

We have already obtained the $(i-1)^{th}$ approximation, ${}^{m+1}\{U\}^{(i-1)}$, to the displacement ${}^{m+1}\{U\}$. Expanding $\Psi({}^{m+1}\{U\})$ using the Taylor series expansion at ${}^{m+1}\{U\}^{(i-1)}$ and neglecting all higher order terms, we get

$$\Psi({}^{m+1}\{U\}^{(i-1)}) + \left.\frac{\partial \Psi}{\partial U}\right|_{{}^{m+1}\{U\}^{(i-1)}} ({}^{m+1}\{U\} - {}^{m+1}\{U\}^{(i-1)}) = 0$$

or

$$\left.\frac{\partial F}{\partial U}\right|_{{}^{m+1}\{U\}^{(i-1)}} \{U\}^{(i)} - {}^{m+1}\{F\}^{(i-1)} - {}^{m+1}\{R\} = 0 \tag{3.45}$$

where

$$\{\Delta U\}^{(i)} = {}^{m+1}\{U\} - {}^{m+1}\{U\}^{(i-1)} \tag{3.46}$$

$${}^{m+1}\{F\}^{(i-1)} = {}^{m+1}\{F ({}^{m+1}\{U\}^{(i-1)}) \} \tag{3.47}$$

Recognizing that

$$^{m+1}[K]^{(i-1)} = \left.\frac{\partial F}{\partial U}\right|_{^{m+1}\{U\}^{(i-1)}}$$

$$= \int_V [B]^T \left.[D^{ep}]\right|_{^{m+1}\{U\}^{(i-1)}} [B]\, dV \tag{3.48}$$

where $[D^{ep}]|_{^{m+1}\{U\}^{(i-1)}}$ is the elastic-plastic stiffness corresponding to displacement $^{m+1}\{U\}^{(i-1)}$, and $^{m+1}[K]^{(i-1)}$ is the tangential stiffness matrix of the structure, we obtain the iteration scheme of the Newton-Raphson algorithm as

$$^{m+1}[K]^{(i-1)}\{\Delta U\}^{(i)} = {^{m+1}\{R\}} - {^{m+1}\{F\}^{(i-1)}} \tag{3.49a}$$

$$^{m+1}\{U\}^{(i)} = {^{m+1}\{U\}^{(i-1)}} + \{\Delta U\}^{(i)} \tag{3.49b}$$

$$^{m+1}\{U\}^{(0)} = {^{m}\{U\}}, \qquad {^{m+1}[K]^{(0)}} = {^{m}[K]},$$

$$^{m+1}\{F\}^{(0)} = {^{m}\{F\}} \qquad (i = 1,2,....) \tag{3.49c}$$

This iteration continues until the proper convergence criterion is satisfied. The iteration procedure is shown schematically in Fig. 3.2.

The Newton-Raphson method converges quadratically at a faster rate. However, it should be noted from Eq. (3.49) that the tangential stiffness matrix , $^{m+1}[K]^{(i-1)}$, is evaluated and factorized at each iteration step , and such an operation can be prohibitively expensive when a large-scale system is considered. Moreover, for perfectly plastic or a strain softening material, the tangential stiffness matrix may become singular or ill-conditioned. This may cause

difficulty in the iteration procedure. Modifications of the Newton-Raphson algorithm are therefore necessary, and will be described in the following section.

3.5.2 Modified Newton-Raphson Method

One of the modification of the Newton-Raphson method is to replace the tangential stiffness matrix ${}^{m+1}[\ K'\ ]^{(i-1)}$ in Eq. (3.49a) by ${}^{n}[\ K\ ]$, which is a tangential stiffness matrix evaluated at load step *n*, *n* < m+1. Matrix ${}^{n}[\ K\ ]$ is evaluated at the beginning of each load step, or for the $(m+1)^{th}$ step, the stiffness matrix

$$ {}^{n}[K] = {}^{m+1}[K]^{(0)} = {}^{m}[K] \tag{3.50}$$

is used. The iteration scheme for the *modified Newton-Raphson* algorithm is expressed as

$$ {}^{n}[K]\ \{\Delta U\}^{(i)} = {}^{m+1}\{R\} - {}^{m+1}\{F\}^{(i-1)} \tag{3.51a}$$

$$ {}^{m+1}\{U\}^{(i)} = {}^{m+1}\{U\}^{(i-1)} + \{\Delta U\}^{(i)} \tag{3.51b}$$

$$ {}^{m+1}\{U\}^{(0)} = {}^{m}\{U\},$$

$$ {}^{m+1}\{F\}^{(0)} = {}^{m}\{F\} \qquad (i = 1,2,....) \tag{3.51c}$$

Again, the iteration continues until a proper convergence criterion is satisfied. This modified iteration procedure is shown schematically in Fig. 3.3.

The modified Newton-Raphson algorithm involves fewer stiffness matrix evaluation and factorization steps. As a result, computational

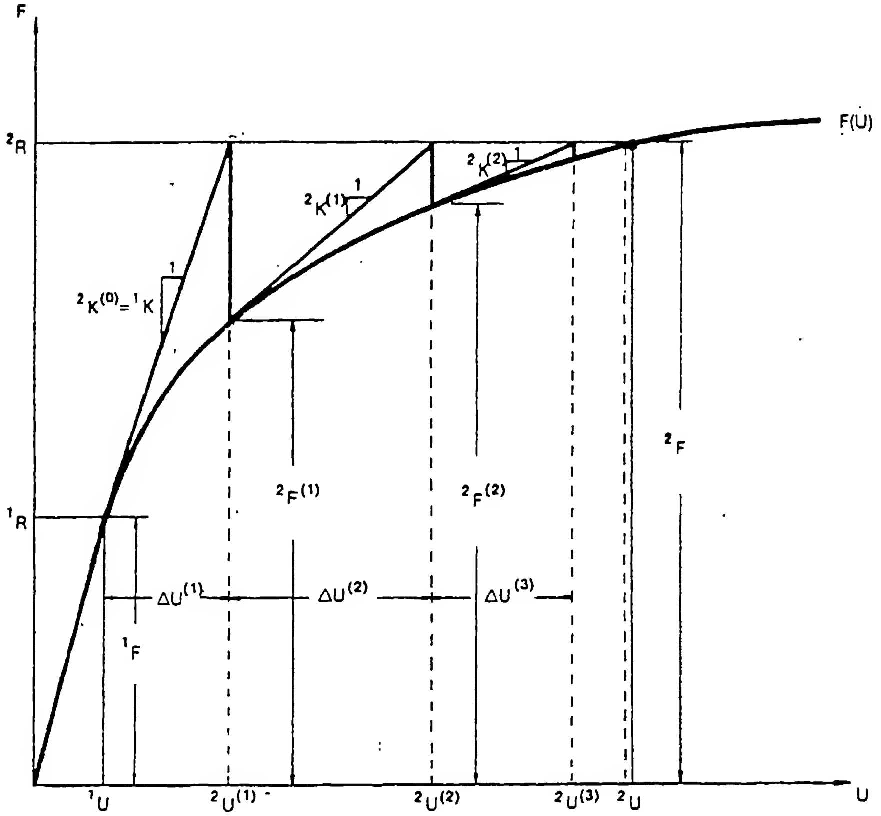


Fig. 3.2 Newton-Raphson Method

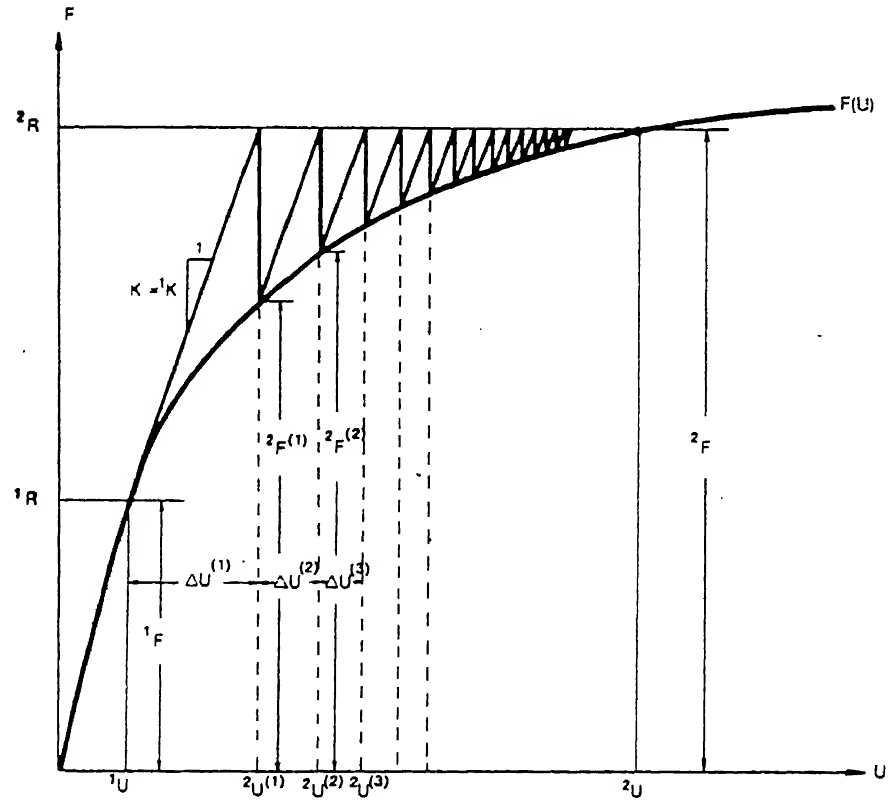


Fig. 3.3 Modified Newton-Raphson Method [7]

effort for one iteration cycle is much less than for the Newton-Raphson algorithm for a large-scale system. However, the modified Newton-Raphson method converges linearly and, in general more slowly than the Newton-Raphson method, i.e., for a given problem, more iterations are needed to reach a convergence when the modified Newton-Raphson method is used. The convergence rate of the modified Newton -Raphson algorithm depends to a large extent on the number of times the stiffness matrix is updated.

Moreover, the problem that the stiffness matrix may be singular or ill-conditioned still exists. Another problem associated with the modified Newton-Raphson method is that if a change in the external load cause an unloading, i.e., the stress state is unloaded from plastic state to elastic state, this method may not lead to converge for that iteration , unless the structural stiffness matrix is updated for this situation. This in turn increases the complexity of coding in the numerical implementation of the modified method.

Convergence Criterion

A properly defined convergence criterion to terminate the equilibrium iteration is an essential part of an efficient incremental solution strategy. At the end of each iteration , the solution obtained must be checked against a selected tolerance to see whether convergence has occurred.

For an equilibrium iteration ,solution $\{ U \}$ is sought at which the difference between the external force and internal force, $\{ {}^{m+1}\{ R \} - {}^{m+1}\{ F \} \}$ is less than or equal to an allowable value. This criterion is called *force criterion*.

The tolerance must be carefully chosen. A loose tolerance gives inaccurate results, while a too tight tolerance may leads to wasteful computation to obtain a needless accuracy.

3.5.3 YAMADA'S METHOD

In this section a step by step incremental algorithm to solve non-linear problems is presented. The sequence of the program is as follows

1> Apply a unit test load or unit displacements and compute the displacement vector(Δu), $\varepsilon^{\bullet}$, $\sigma^{\bullet}$ and $\bar{\sigma}$

2> Find the element which has maximum $\bar{\sigma}$(say for j^{th} element). Let $\bar{\sigma}_{max}$ be the maximum $\bar{\sigma}$ of all the elements. calculate a scale factor according to $r_{\bullet} = Y/\bar{\sigma}_{max}$

3> Multiply Δu, ε, σ, $\bar{\sigma}$ and load (ΔF) by the scale factor $r_{\bullet}$ to obtain the values corresponding to the load that is required to just yield one element. The j^{th} element (i.e. the element which is having maximum stress) is considered as a post yielded element from the next computation.

4> Add Δu to the initial coordinates of the nodal points in order to follow the path of deformation.

5> Calculate elastic-plastic stiffness matrix $[K^{ep}]$ for all the yielded elements using $[D^{ep}]$ given in section 3.2. Replace the elastic stiffness matrices with the above calculated $[K^{ep}]$ for the yielded elements in the global stiffness matrix.

6> Apply an incremental load or displacement and compute the corresponding values of Δu, $\Delta\varepsilon$ and $\Delta\sigma$.

7> By making use of the following equation calculate r for every element remaining in the elastic state.

$$r = \frac{\Gamma + \sqrt{\Gamma^2 + 4\,(\Delta\bar{\sigma}_{step})^2\,(Y^2 - \bar{\sigma}^2)}}{2(\Delta\bar{\sigma}_{step})^2} \qquad (3.52)$$

Derivation for r is given in the appendix -II

8) Find the element which is having minimum r (say for k^{th} element) and let r_1 be the r for that element. The load increment multiplied by the factor r_1 is just sufficient to yield the k^{th} element. From the next iteration k^{th} element is considered to be yielded.

9) Multiply Δu, $\Delta\varepsilon$, $\Delta\sigma$ and load increment by the factor r_1 and add to the present displacements, strains, stresses and load. Add Δu to the coordinates of the nodal points.

10) Compute $\bar{\sigma}$ for each element.

11) Check $\bar{\sigma}$ of any element remaining in the elastic state has reached 0.995Y. The element having $\bar{\sigma}$ is equal to or greater than 0.995Y is treated as yielded element in the next step and thereafter. In doing so total number of iterations will be reduced.

12) Calculate the increment $\Delta\bar{\varepsilon}^p$ of equivalent plastic strain for each post yielded element using

$$\Delta\bar{\varepsilon}^{\,p} = \frac{\sigma'_{ij}\;\Delta\varepsilon_{ij}}{\bar{\sigma}\left[1 + \dfrac{H'}{3G}\right]}$$

13) Check that $\Delta\bar{\varepsilon}^p$ is positive for all the post yielded elements. If positive, go to the step <5> and repeat the process otherwise stop doing the process.

$\Delta\bar{\varepsilon}^p < 0$ means the plastic region of the non hardening material

expands to such an extent that large plastic strains are possible without any practical increase of load. Though the implementation of the $\Delta\bar{\varepsilon}^{p}$ has not been fully understood, it can be concluded that it is an indication that we approach very close to the collapse load.

CHAPTER 4

FORMULATION OF THE EXTRUSION PROBLEM

This chapter deals with *Elastic-Plastic analysis* of extrusion process .The following two cases are dealt with

1. Transient analysis.
2. Static analysis.

In transient analysis the die is not completely filled with material as opposed to the static analysis in which the die is completely filled with the material before starting the process. These cases are analysed for both plane-strain and axisymmetric problems.

The flow sequence of the program and the expressions for total work done, plastic work done, frictional losses and the normal die reaction forces are discussed in detail.

The total work done is calculated by using the following relation

$$W^t = \int_V \int_o^{\varepsilon_{ij}} \sigma_{ij} \, d\varepsilon_{ij} \, dV \qquad (4.1)$$

Where

v is the volume of the domain

$$d\varepsilon_{ij} = d\varepsilon^e_{ij} + d\varepsilon^p_{ij} \qquad (4.2)$$

The elastic work done is given by

$$W^e = \int_V \int_o^{\varepsilon^e_{ij}} \sigma_{ij} \, d\varepsilon^e_{ij} \, dV \qquad (4.3)$$

The plastic work done (W^p) is the difference of the total work

done and elastic work done

Plastic work done and elastic work done are shown in figure 4.1 at a particular state of stress.

4.1. TRANSIENT STUDY

The figure 4.2 shows the billet and the punch positions. The billet is forged at the end to fit into the die before the extrusion process takes place. The discretization of the material is shown in fig. 4.3. The symmetry about x-axis is exploited to reduce the computational cost. A finer mesh is taken near the die surface to get good results.The domain is discretized into triangular elements.

A computational algorithm for the incremental displacement method using the elasto-plastic analysis for the above problem is presented here.

1> Apply a unit punch displacement and compute elastic displacements at the nodes {u} , forces at the nodes {F}, strains $\{\varepsilon\}$, and stresses $\{\sigma\}$ at each element.

2> Calculate the equivalent stresses $\bar{\sigma}$ and equivalent strains $\bar{\varepsilon}$ of each element.

3> Find the element which has the maximum $\bar{\sigma}$ (say for j^{th} element). Let $\bar{\sigma}_{max}$ be the maximum $\bar{\sigma}$ of all the elements. Calculate a scale factor $r_e = \dfrac{Y}{\bar{\sigma}_{max}}$,where Y denotes the yield stress of the material under consideration.

The following figure shows the factor r_e graphically.

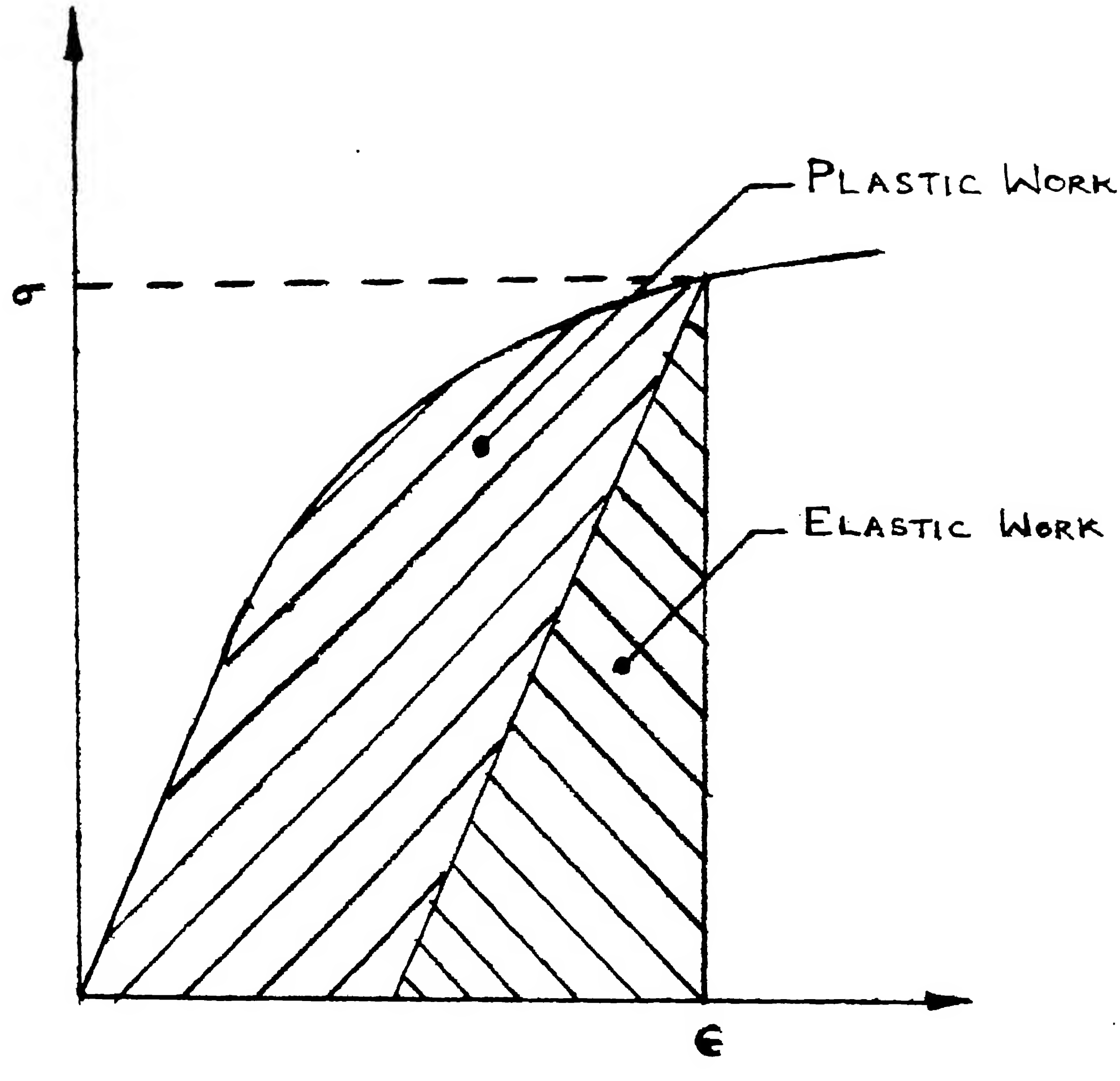


Fig. 4.1 Elastic Work and Plastic Work

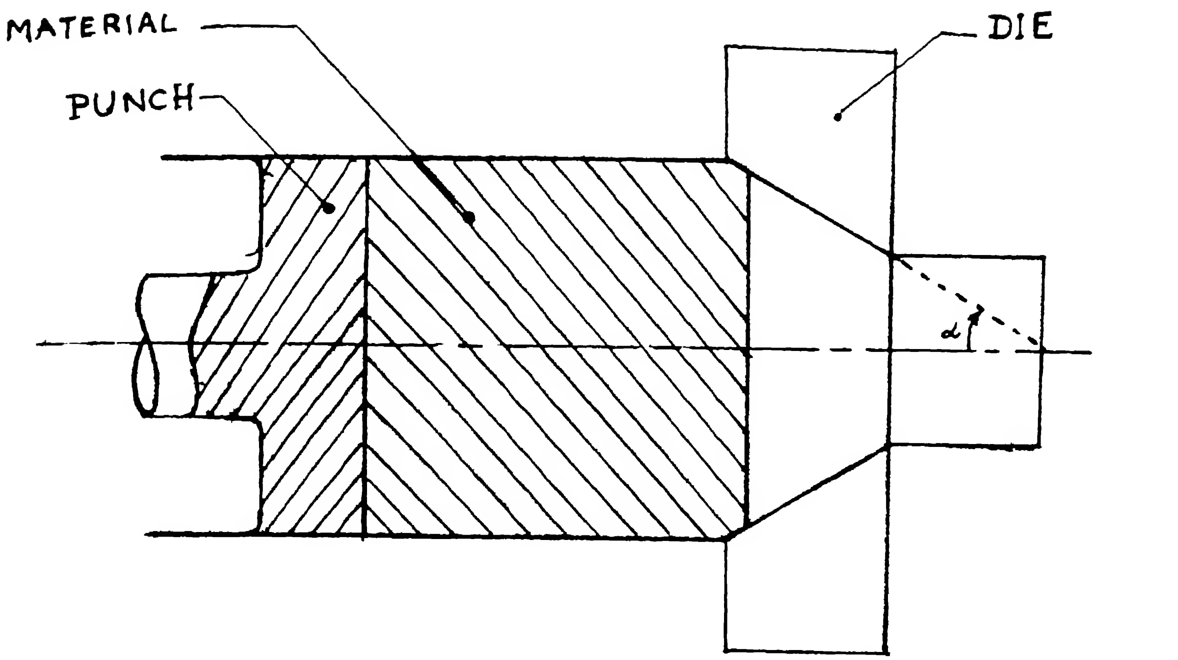


Fig. 4.2 The position of material, punch and die when the die is not completely fill (transient case)

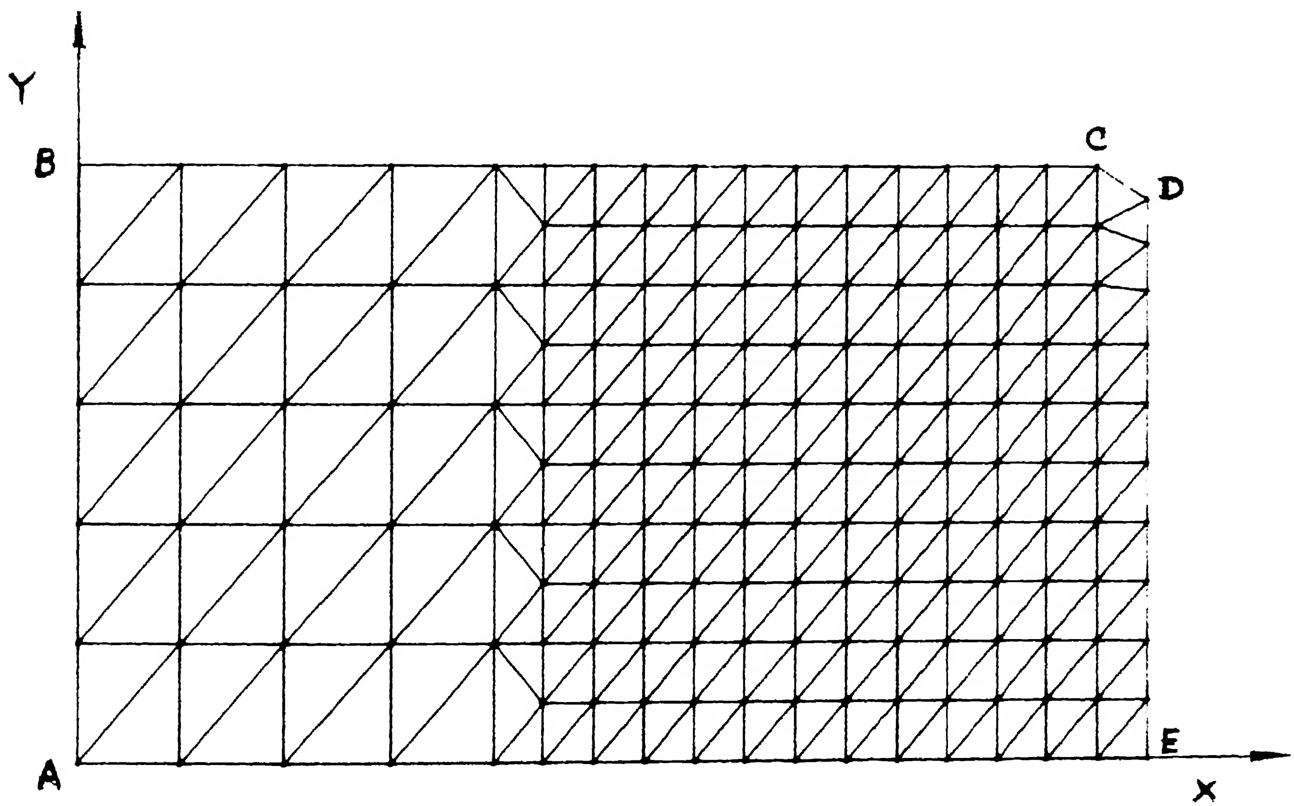
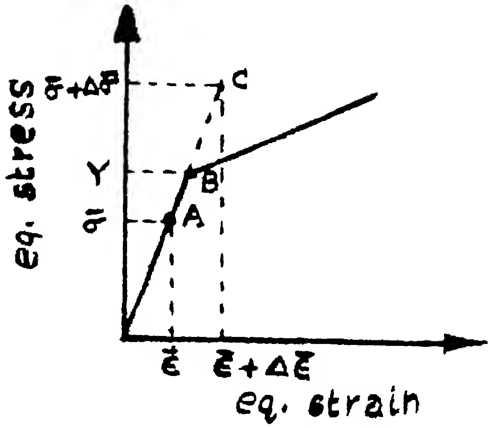


Fig. 4.3 Mesh for the F.E.M. analysis for transient case.



$$r_e = \frac{AB}{AC}$$

4) Multiply {u}, {F}, {σ}, {ε}, $\bar{\sigma}$, and $\bar{\varepsilon}$ by the scale factor r_e to obtain the values corresponding to those at the elastic limits. The j^{th} element (the element which is having $\bar{\sigma}_{max}$) is treated as an yielded element for the next computation.

5) Add {u} to the initial co-ordinates of the nodal points in order to follow the path of deformation.

6) Calculate $[D^{ep}]$ and hence forth $[K^{ep}]$ for all the post-yielded elements or continued loaded elements $[D^{ep}]$ is given in chapter 3. Replace the stiffness matrix for the yielded elements with the above corresponding plastic stiffness matrix.

7) Apply the incremental punch displacement and compute the corresponding values of {Δu}, {ΔF}, {$\Delta\varepsilon$}, {$\Delta\sigma$}.

8) By making use of the following equation calculate r for all the elements those are remaining in the elastic state.

$$r = \frac{\Gamma + \sqrt{\Gamma^2 + 4\,(\Delta\bar{\sigma}_{step})^2\,(Y^2 - \bar{\sigma}^2)}}{2(\Delta\bar{\sigma}_{step})^2}$$

Find the minimum r (say for m^{th} element r is minimum). Let the minimum r be r_1

9) Calculate r_2 for the node which is nearer (say i^{th} node) to the entrance of the die and on the container surface using the following relation.

$$r_2 = \frac{(x_{dieent} - x_i)}{\Delta u_{xi}} \tag{4.4}$$

Where

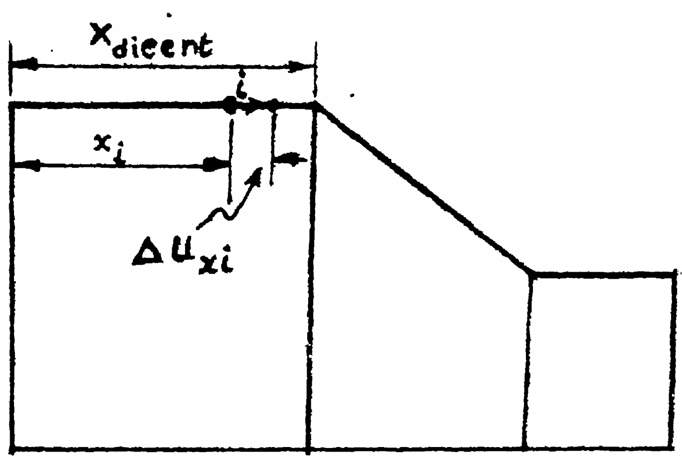
x_{dieent} is the x-coordinate of the die entrance.

x_i is the x-coordinate of the node which is nearer to the die entrance.

Δu_{xi} is the displacement in x-direction of i^{th} due to the incremental punch displacement.

r_2 determines the punch displacement that is just sufficient for the i^{th} node to reach the die entrance.

The following figure shows the notations used to calculate r_2



10) Calculate r_c of all the possible nodes which may come in contact with the die surface due to the incremental punch displacement by making use of the following relations.

$$Y_1 = Y_{dieent} - \tan(\alpha)\,(x_j - x_{dieent}) \qquad (4\cdot5a)$$

$$Y_2 = Y_{dieent} - \tan(\alpha)\,(x_j + du_{xj} - x_{dieent}) . \qquad (4\cdot5b)$$

$$a = Y_1 - y_j \qquad (4\cdot5c)$$

$$b = Y_j + \Delta u_{yj} - Y_2 \qquad (4\cdot5d)$$

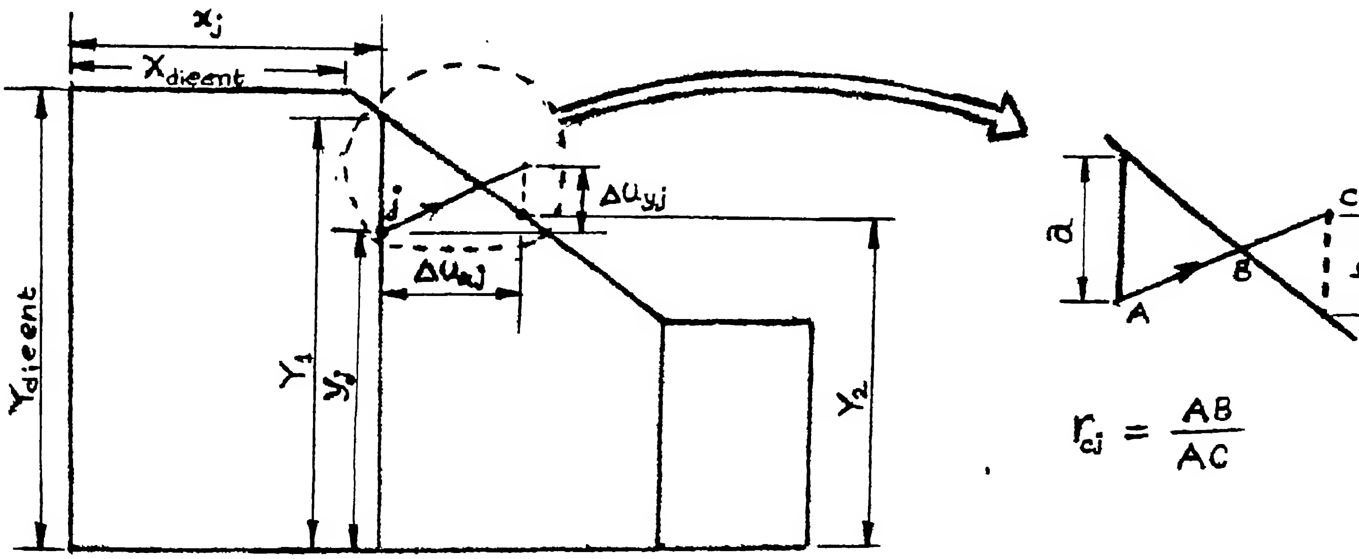
$$r_{cj} = \frac{a}{a + b} \qquad (4\cdot5e)$$

r_{cj} indicates the factor r_c of the j^{th} node.

Where

Y_{dieent} is the inlet radius for axdsymmetric case or half of the inlet thickness of the billet for plane-strain problem.

The following figure shows the all notations used to calculate r_{cj}



Let the minimum(r_{cj}) be r_3(say r_c is minimum for k^{th} node).

11) Let the minimum of r_1, r_2 and r_3 be r_{min} .

12) Multiply {Δu}, {ΔF}, {Δσ} and {Δε} with the factor r_{min}.

13> Add $\{\Delta u\}$, $\{\Delta F\}$, $\{\Delta \sigma\}$, $\{\Delta \varepsilon\}$ to the previous state $\{u\}$, $\{F\}$, $\{\sigma\}$, $\{\varepsilon\}$.

14> Update the co-ordinates of all the nodes.

15> Calculate the equivalent stress and the equivalent strains of all the elements.

16> Calculate the incremental total work done dW^t, elastic work done W^e. Add dW^t to W^t to get the total W^t, and then calculate W^p using the relations given at the beginning of this chapter.

17> conditions to be imposed from the next iteration

(1). If r_1 is r_{min} then,

m^{th} element(i.e. the element for which r is minimum) is considered as the post-yielded element from the next computation

(2). If r_2 is r_{min} then,

j^{th} node (which is nearer to the die entrance and on the container surface) is reached the die entrance with this punch displacement. From the next iteration j^{th} node is considered to move along the die surface CD.

(3). If r_3 is r_{min} then,

k^{th} node (which is having minimum r_c) will come in contact with this punch displacement. From the next iteration k^{th} node is considered to move along the die surface CD.

18> Check if the displacement of the punch is reached the specified displacement then stop doing the process, otherwise go to the step (6) and continue the process till the above condition is satisfied.

BOUNDARY CONDITIONS:

The boundary conditions for both the plane strain and axisymmetric

cases along the surfaces AB, BC, CD, DE, EA are assumed to be as follows in regard to the nodal forces (F_x, F_y) and the nodal displacements (u_x, u_y).

AB: u_x = unit punch displacement and $F_y = 0$;

BC: $F_x = 0$ and $u_y = 0$;

CD: displacement normal to the die surface is zero

i.e. $$u_x \tan(\alpha) + u_y = 0$$

and

We know that $F_t = \mu F_n$

$$F_t = F_x \cos(\alpha) - F_y \sin(\alpha)$$

$$F_n = F_x \sin(\alpha) + F_y \cos(\alpha)$$

$$\therefore \quad F_x(\mu \tan(\alpha) - 1) + F_y(\mu + \tan(\alpha)) = 0$$

It is assumed that along the die surface there is no sticking and only slip occurs accordingly condition is given by

$$F_x(\mu \tan(\alpha) - 1) + F_y(\mu + \tan(\alpha)) = 0$$

DE: $F_x = 0$ and $F_y = 0$;

EA: $F_x = 0$ and $u_y = 0$;

4.2. STATIC ANALYSIS

In this case initially, the die is completely filled with the material and is assumed to be in the elastic state. Figure 4.4 shows the billet position in the die. Making use of the symmetry about the x-axis only half of the portion is analysed to reduce the computational effort. The mesh for this case is shown in figure 4.5

The computational procedure is same as that of the transient case, except for the calculation of r_3

In this case r_3 is calculated as follows

$$r_3 = \frac{x_{dieext} - x_i}{\Delta u_{xi}} \qquad (4 \cdot 6)$$

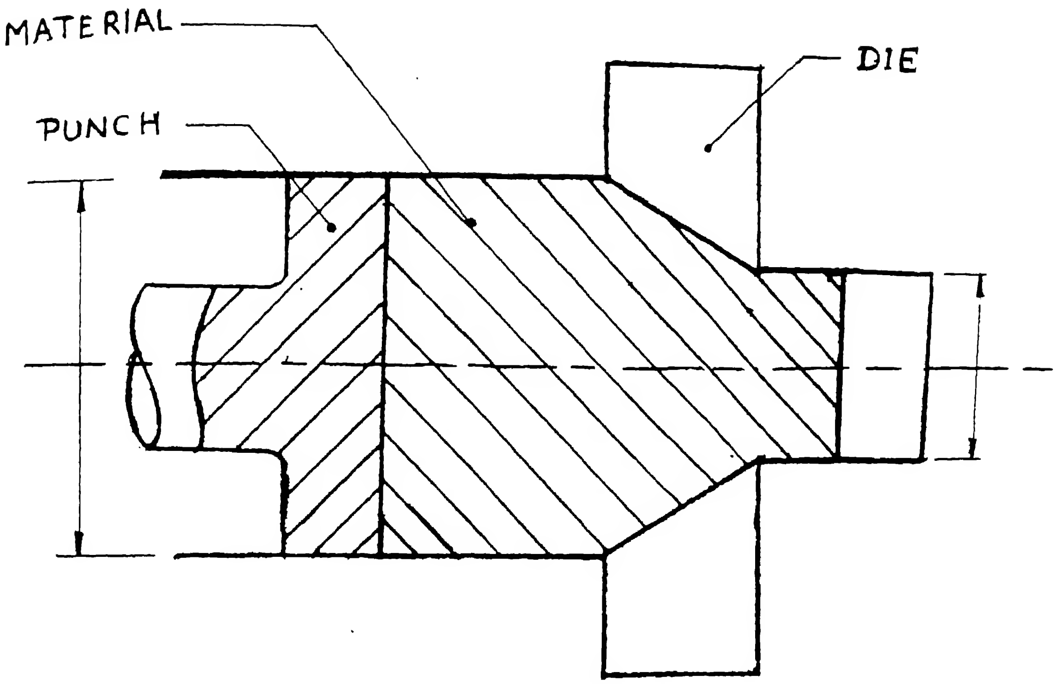


Fig. 4.4 The position of material, punch and die when the die is completely fill (static case)

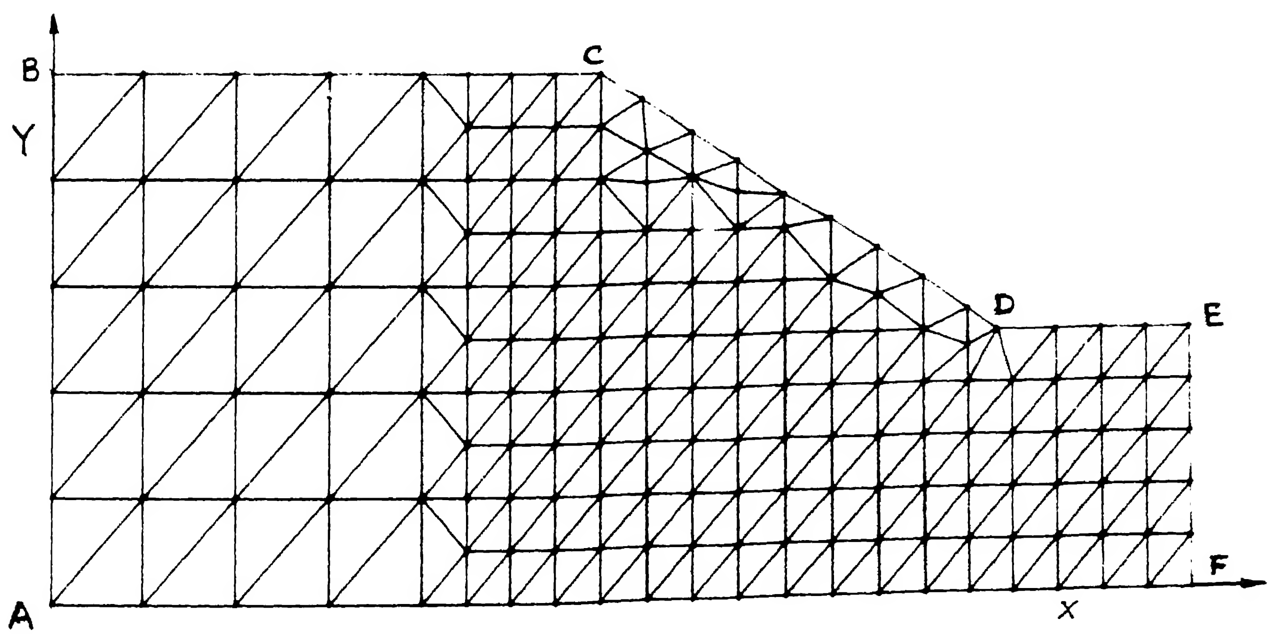


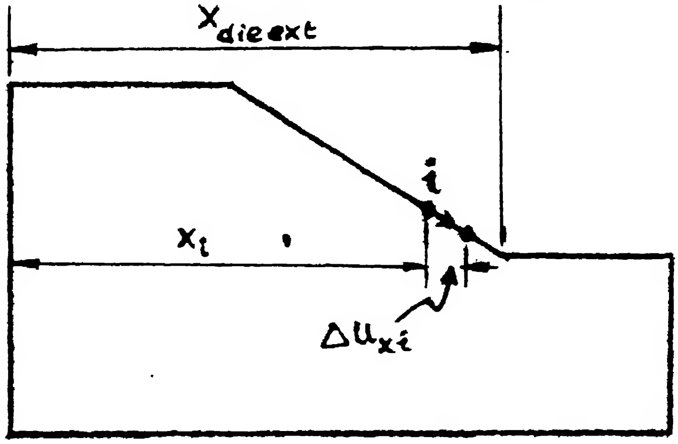
Fig. 4.5 Mesh for the F.E.M. analysis for static case

Where

x_{dieext} is the x-coordinate of the exit of the die

x_l is the x-coordinate of the node which is nearer to the die exit and is on the die surface.

All the above notations are shown in the figure given bolow.



Here r_3 represents the factor that should give for the punch incremental displacement to just move the i^{th} node (which is on the surface of the die and nearer to the die exit) to the die exit.

If r_3 is minimum of r_1, r_2 and r_3 then i^{th} node should move along DE as shown in figure 4.5 from the next iteration.

BOUNDARY CONDITIONS

AB: u_x = specified punch displacements and $F_y = 0$;

BC: $u_y = 0$ and $F_x = 0$;

CD: $u_x \tan(\alpha) + u_y = 0$

and

$$F_x(\mu\tan(\alpha) - 1) + F_y(\mu + \tan(\alpha)) = 0 ;$$

DE: $u_y = 0$ and $F_x = 0$;

EF: $F_x = 0$ and $F_y = 0$;

FA: $u_y = 0$ and $F_x = 0$;

CHAPTER 5

RESULTS AND DISCUSSIONS

In this chapter results for a V-notched problem are discussed and compared with the Yamada's[1] results to validate the program and also results for the extrusion problem are presented.

5.1 V-NOTCHED PROBLEM

Fig. 5.1 gives the details of the v-notched tension specimen with $\alpha = 90^{\circ}$ (notch angle). The material is assumed to be elastic-perfectly plastic and elastic linearly work hardening material. The material properties are as follows.

Young's modulus	E	$= 20000\ N/mm^2$
Poisson's ratio	ν	$= 0.3$
Yield stress	Y	$= 30\ N/mm^2$
Strain hardening parameter	H'	$= 0\ N/mm^2$ (for perfectly plastic)
		$= 1000\ N/mm^2$ (for bilinear material)

P represents the load per half width of the specimen. A_y is the minimum cross sectional area of the specimen. By making use of the symmetry about X-axis and Y-axis one quadrant of the specimen is analysed to reduce the computational cost. The discretization of the specimen is shown in fig. 5.2

Fig. 5.3 shows the development of the plastic zone at different stages of iterations. Numbers in the uppermost left corner represent the relevant stage number.

Fig. 5.4 shows the variation of load(P) with respect to

deflection(δ) for the V-notched specimen. Deflection(δ) is the deflection corresponding to the point C on the X-axis as shown in fig. 5.2. It is observed that there is no sudden change in the overall stiffness of the specimen at the early stages. The load deflection curve has a sharp bend after certain stage, when the plastic zone, spreading inward from both sides of the notch, meet first at a point on the longitudinal axis (X-axis). After that plastic zone spreads very rapidly on both the sides. The specimen will fail at a stage, when all the element across the cross section (i.e. the element along the Y-axis) are yielded.

Fig. 5.5 shows the distribution of the equivalent stress ($\bar{\sigma}$) contours. From fig. 5.5 it is observed that the region near the root of the notch is highly stress concentrated.

Fig. 5.6 shows the distribution of the longitudinal stress(σ_x) over the minimum section.

Fig. 5.7 shows the characteristics of the materials under consideration.

The following section compares the results with Yamada[1].

TABLE 5.1

Comparison of the results with those published by Yamada [1]

	Present Method	Yamada's Method
Machine	HP 9000 - 800 series	HITAC 5020E
No. of Nodes	398	144
No. of elements	725	245
Mesh pattern	Hexagonal	Hexagonal
No. of stages	43	51
Total computation time	22 min	29 min
P_e	234.6 N	221.643 N
P_e / A_y	0.391	0.369
P_c	723.316 N	734.521 N
P_c / A_y	1.205	1.224
P_c / P_e	3.0897	3.314

5.2 EXTRUSION PROBLEMS

The results for the following problems are presented in this section.

<1> Transient state analysis of plane-strain extrusion problem

<2> Transient state analysis of axisymmetric extrusion problem

<3> Static analysis of plane-strain extrusion problem

<4> Static analysis of axisymmetric extrusion problem

The properties of the material under consideration are as follows

Young's modulus	$E = 12500\ N/mm^2$
Poisson's ratio	$\nu = 0.34$
Yield stress	$Y = 8\ N/mm^2$
Strain hardening coefficient	$H' = 0\ N/mm^2$

The deformed shapes for the different problems are shown in figures 5.8, 5.9 and 5.10. The dotted pattern shows the original shape of the material before going to deform.

The plastic zone spreading for the different problems at different stages are shown in figures 5.11, 5.12, 5.13 and 5.14. The numbers below the each of the figure indicates the number of elements in the plastic zone at that stage.

The contours of the equivalent stress ($\bar{\sigma}$) and equivalent strain ($\bar{\varepsilon}$) for different problems are shown in figures 5.15 to 5.22. From the above figures it is observed that stress is highly concentrated at the die corners.

And finally Fig. 5.23 shows the variations of total-work, elastic-work and plastic-work with displacement of the punch. From Fig. 5.23 it is observed that elastic work is not negligible compared to the total work as in the case of visco-plastic model.

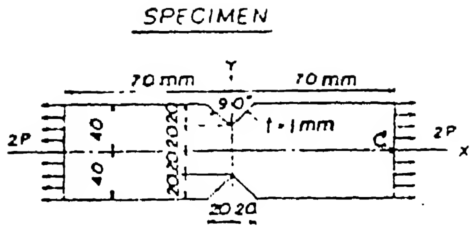


Fig. 5.1 V - notched tension specimen

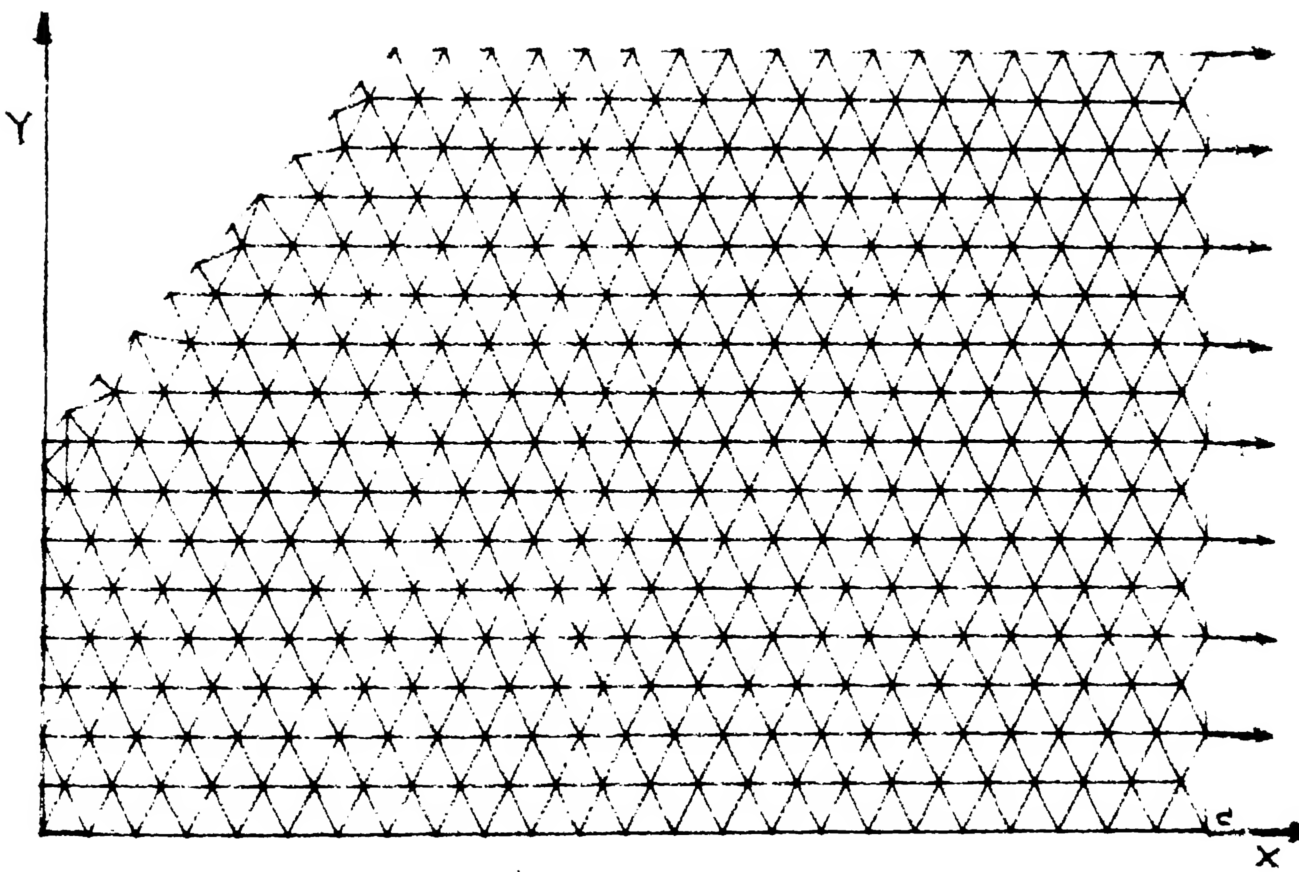
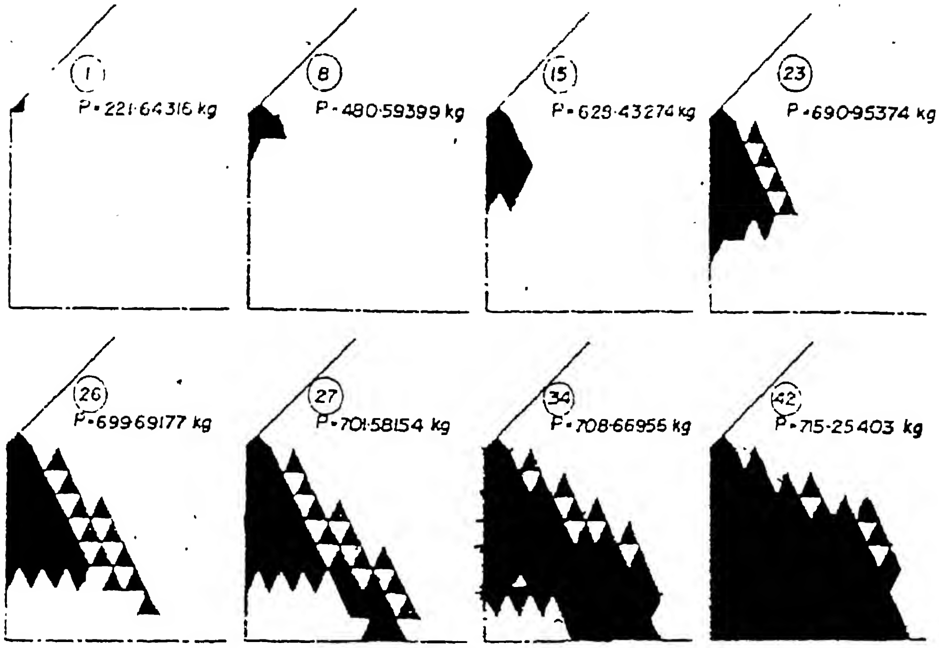
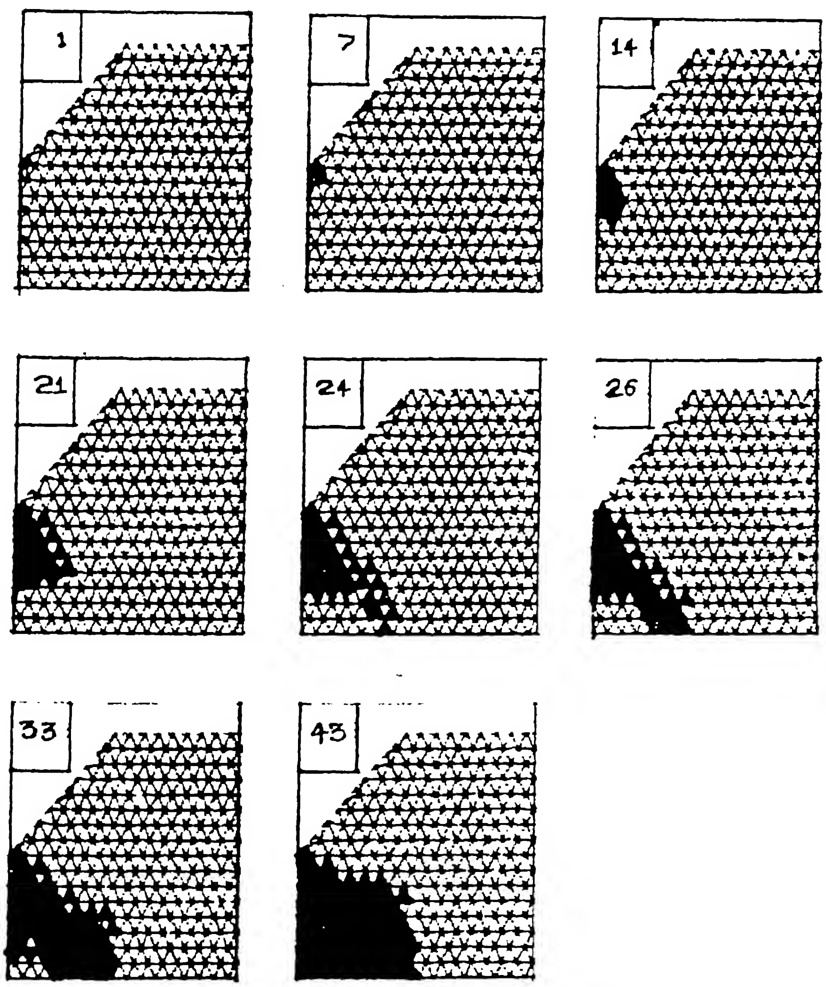


Fig. 5.2 Mesh for the V-notched problem

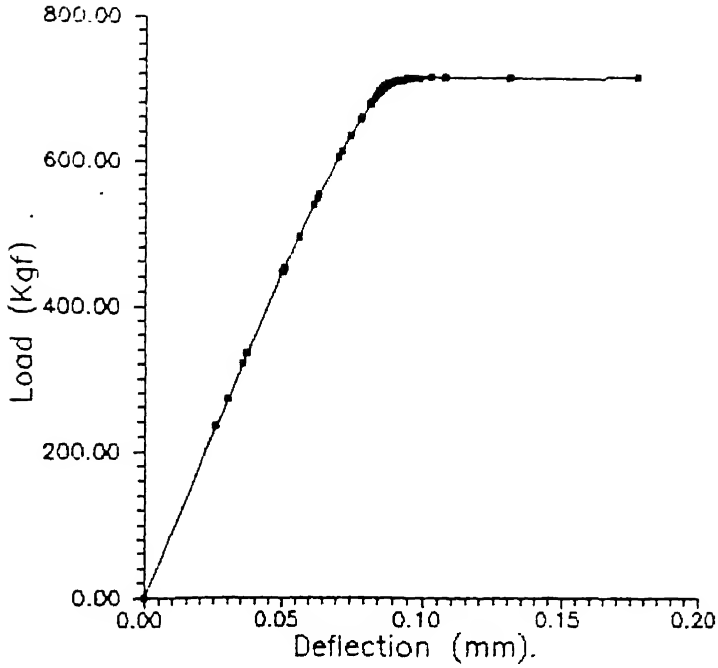


(Yamada s method)

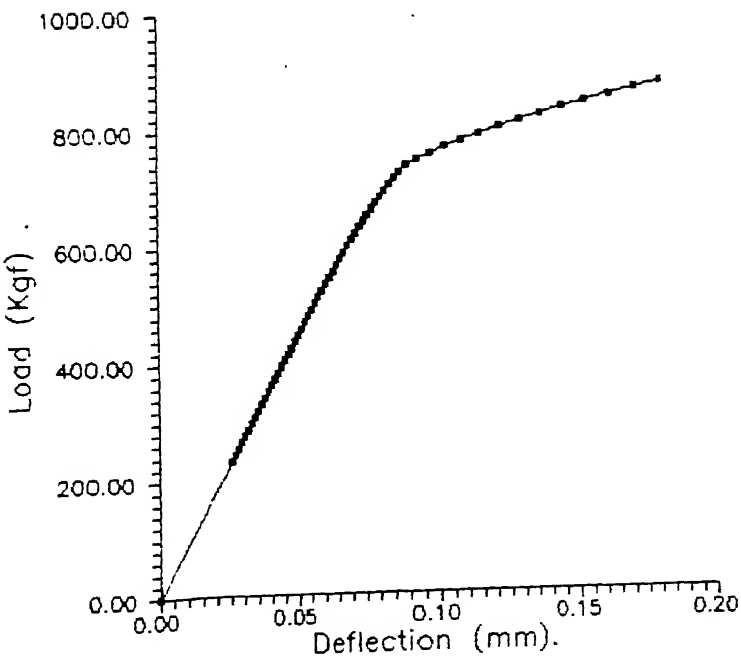


(Present Method)

Fig. 5.3 Development of plastic zone for the V-notch specimen



(H = 0.0)



(H = 1000.0)

Fig. 5.4 Load - Deflection curve for V-notched tension specimen

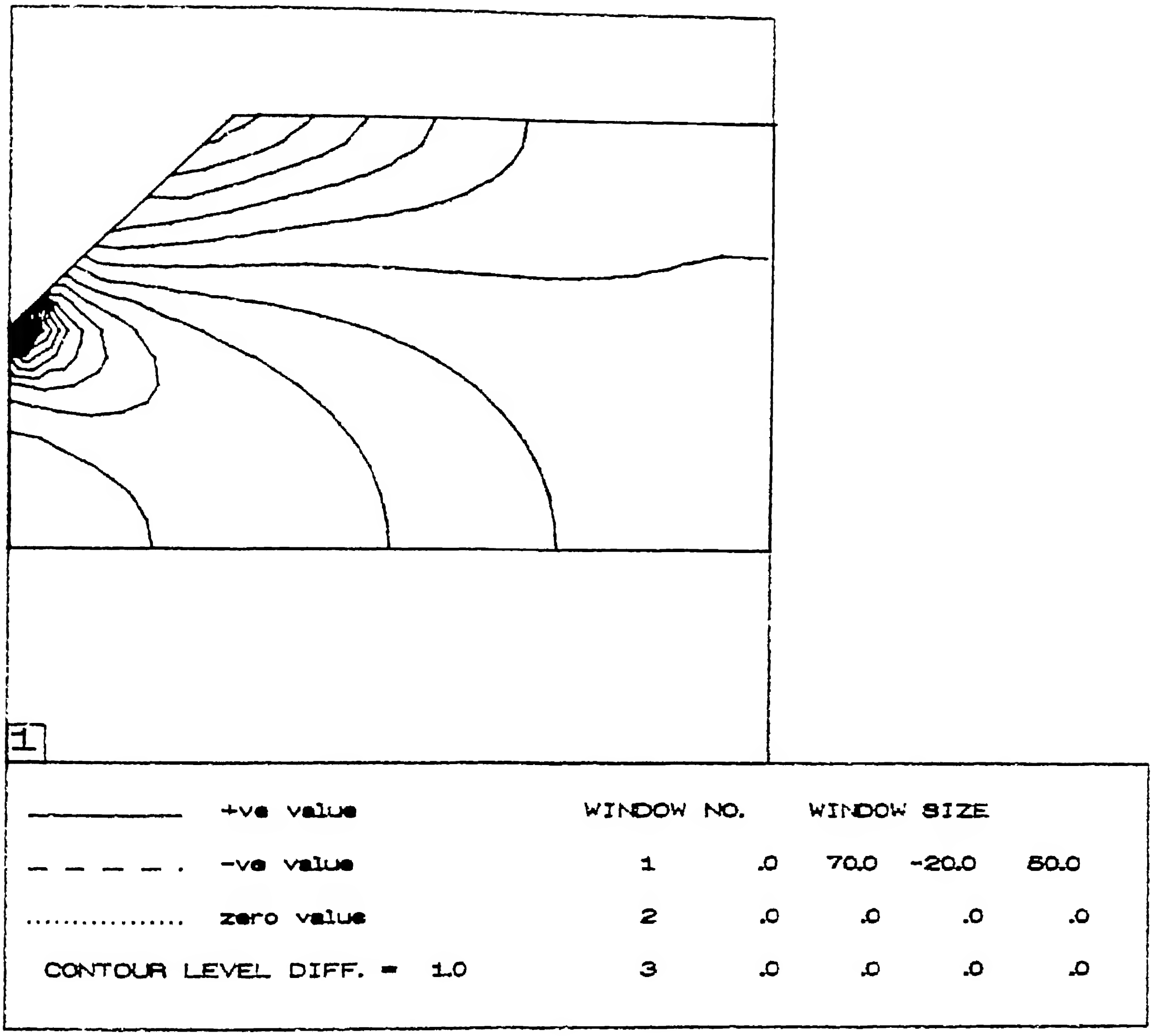
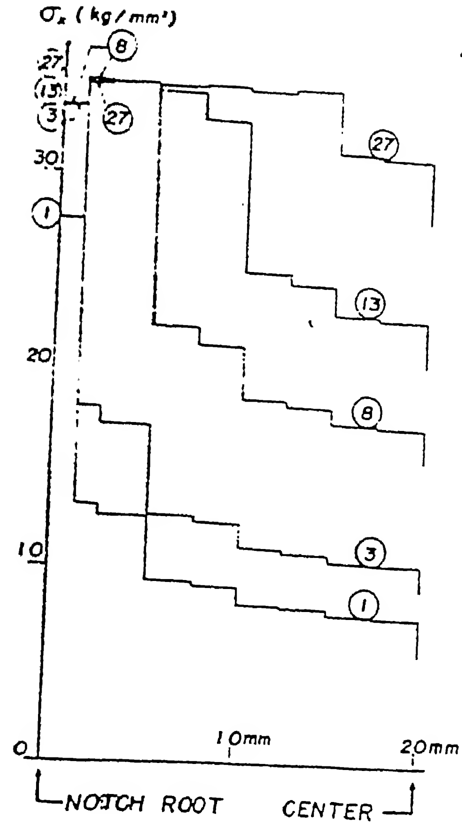
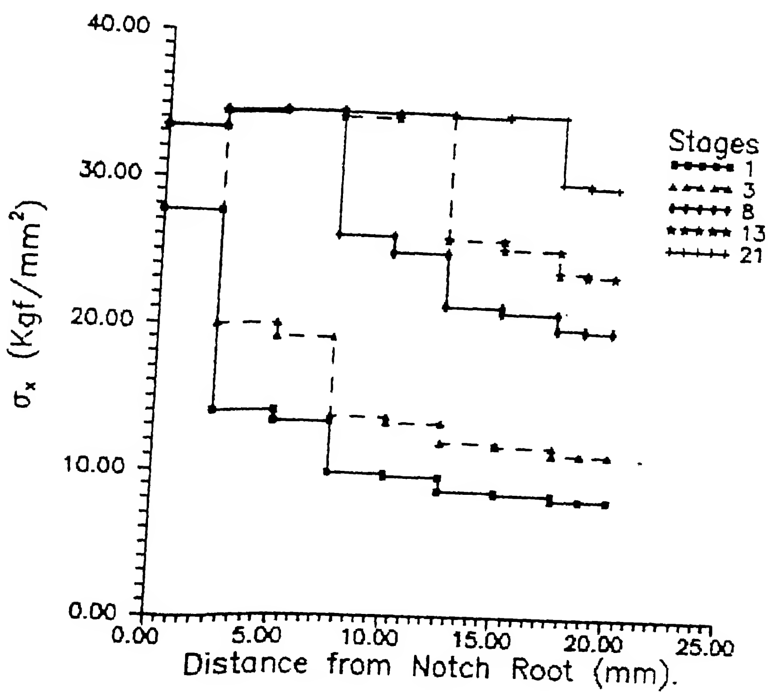


Fig. 5.5 Iso - equivalent stress contours for V-notched specimen

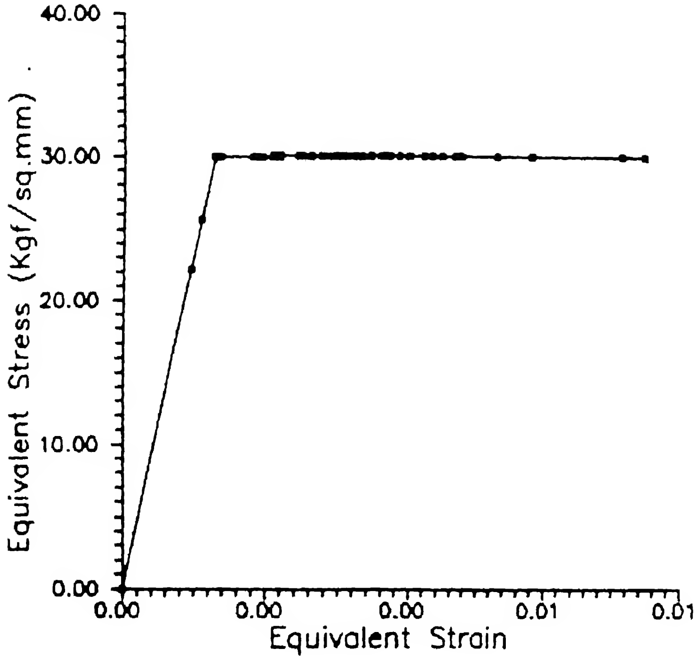


(Yamada's method)

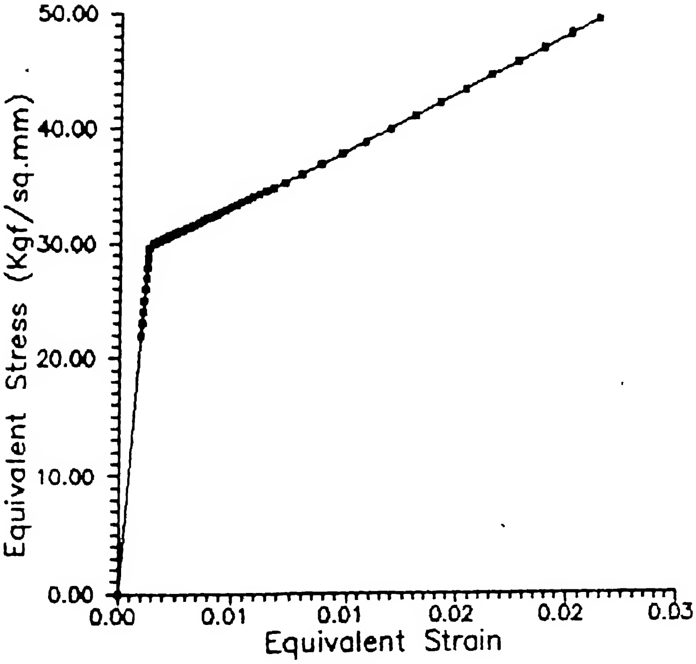


(Present Method)

Fig. 5.6 Distribution of longitudinal stress across the minimum section



(H = 0.0)



(H = 1000.0)

Fig. 5.7 Equivalent stress - Equivalent strain curve for V-notched specimen

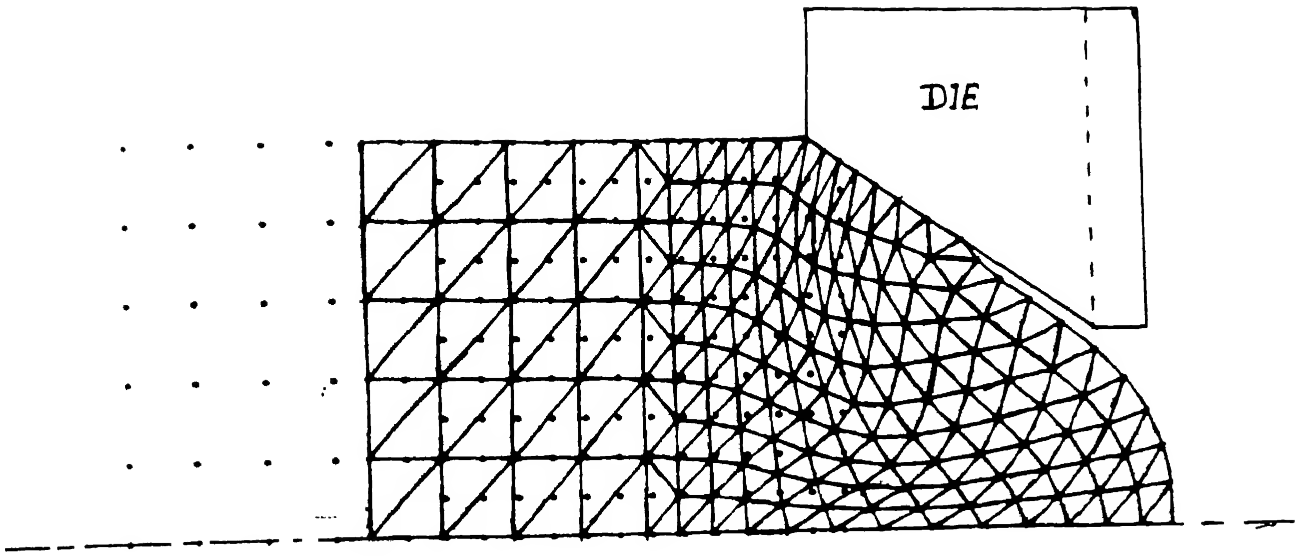


Fig. 5.8 Deformed mesh for transient plane-strain analysis

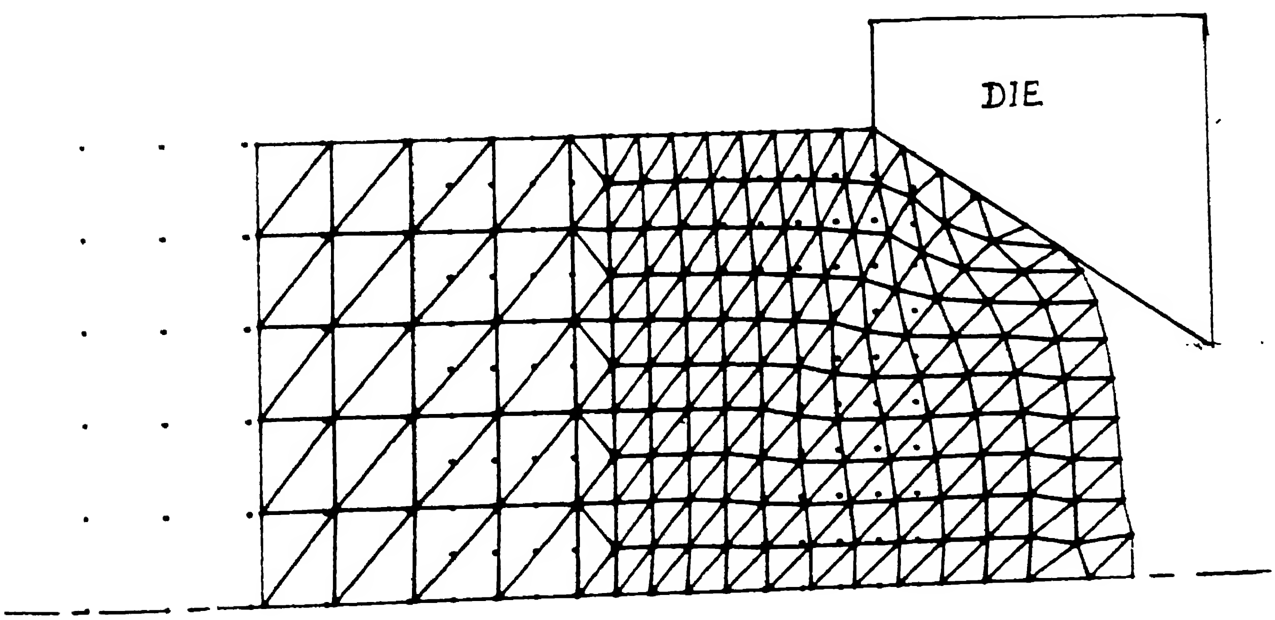


Fig. 5.9 Deformed mesh for transient axisymmetric analysis

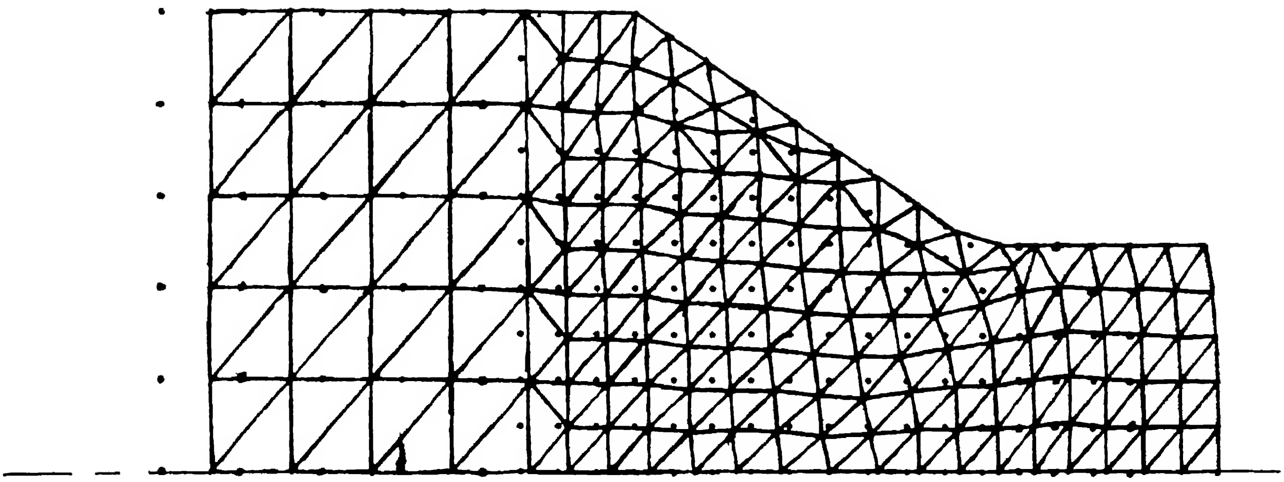


Fig. 5.10 Deformed mesh for static plane-strain analysis

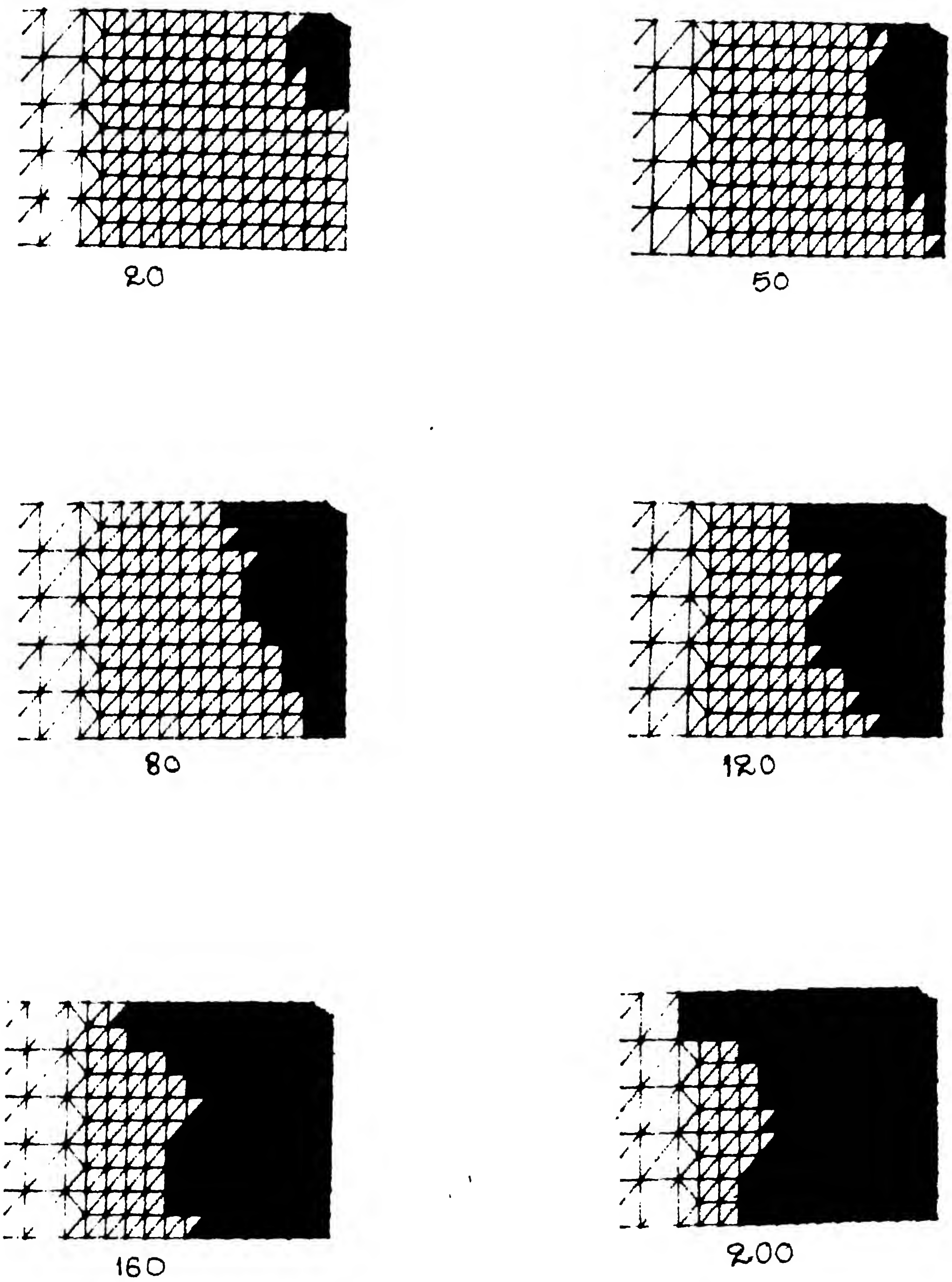


Fig. 5.11 Spreading of plastic zone for transient plane-strain case

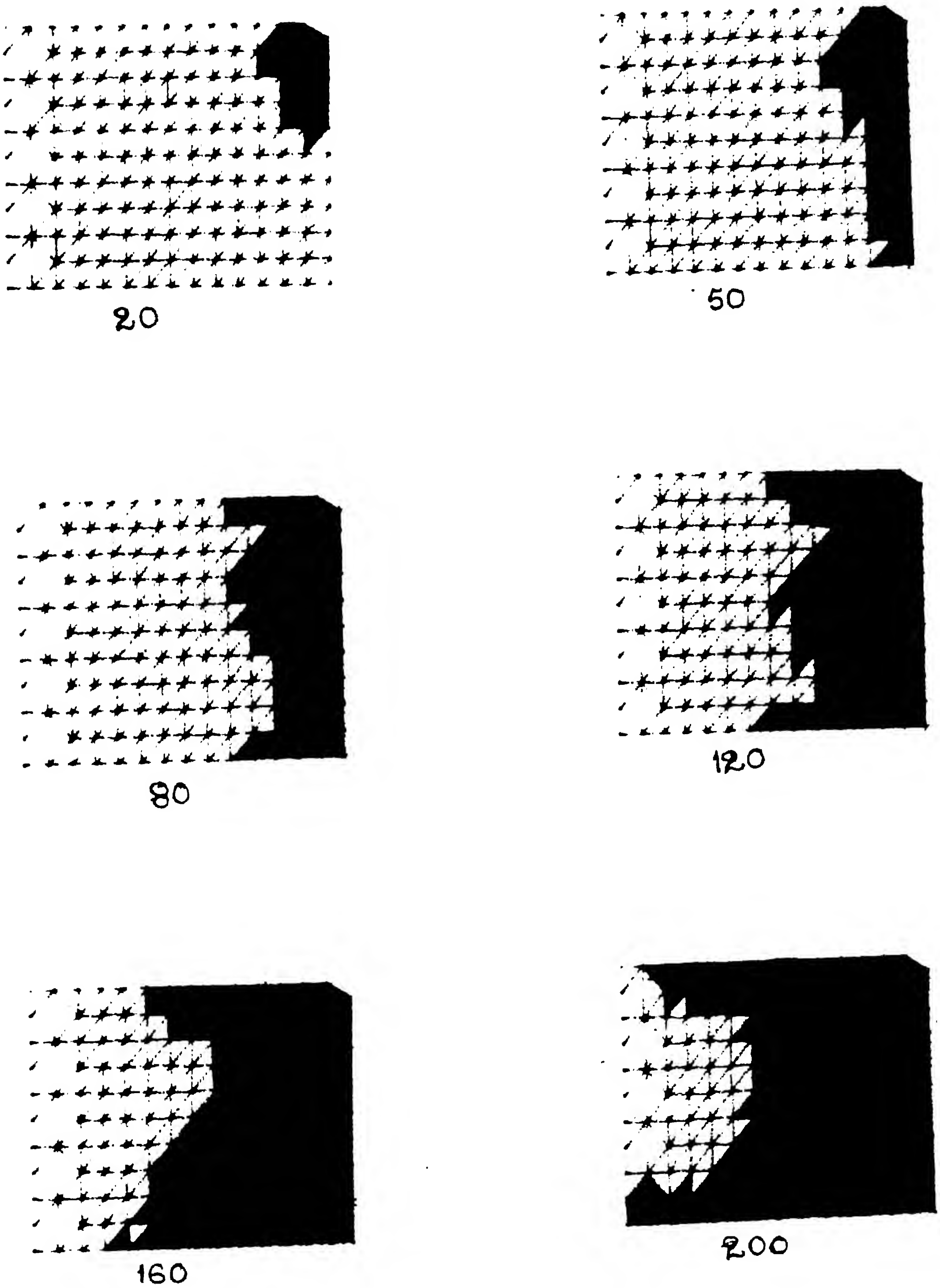


Fig. 5.12 Spreading of plastic zone for transient axisymmetric case

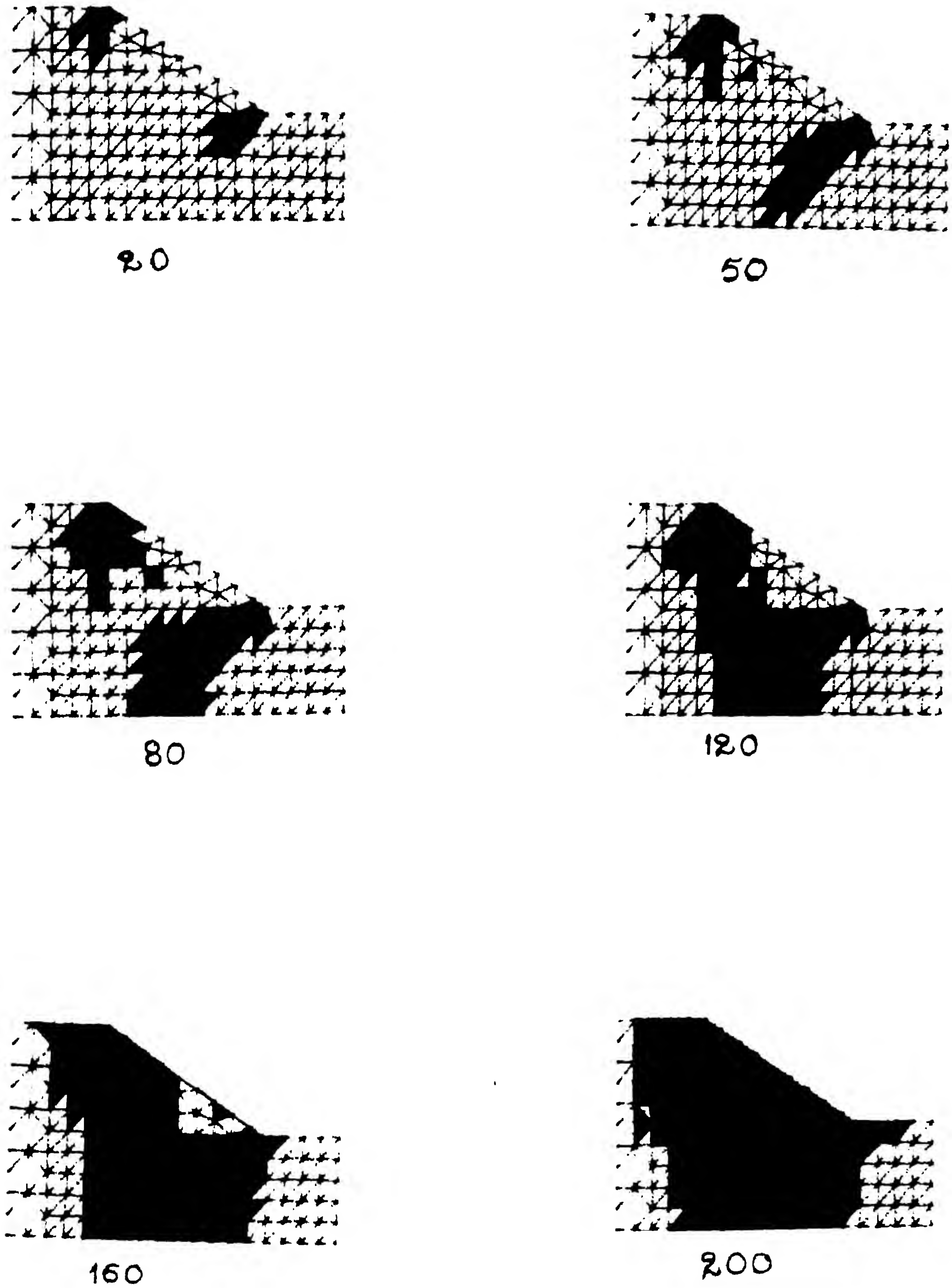


Fig. 5.13 Spreading of plastic zone for static plane-strain case

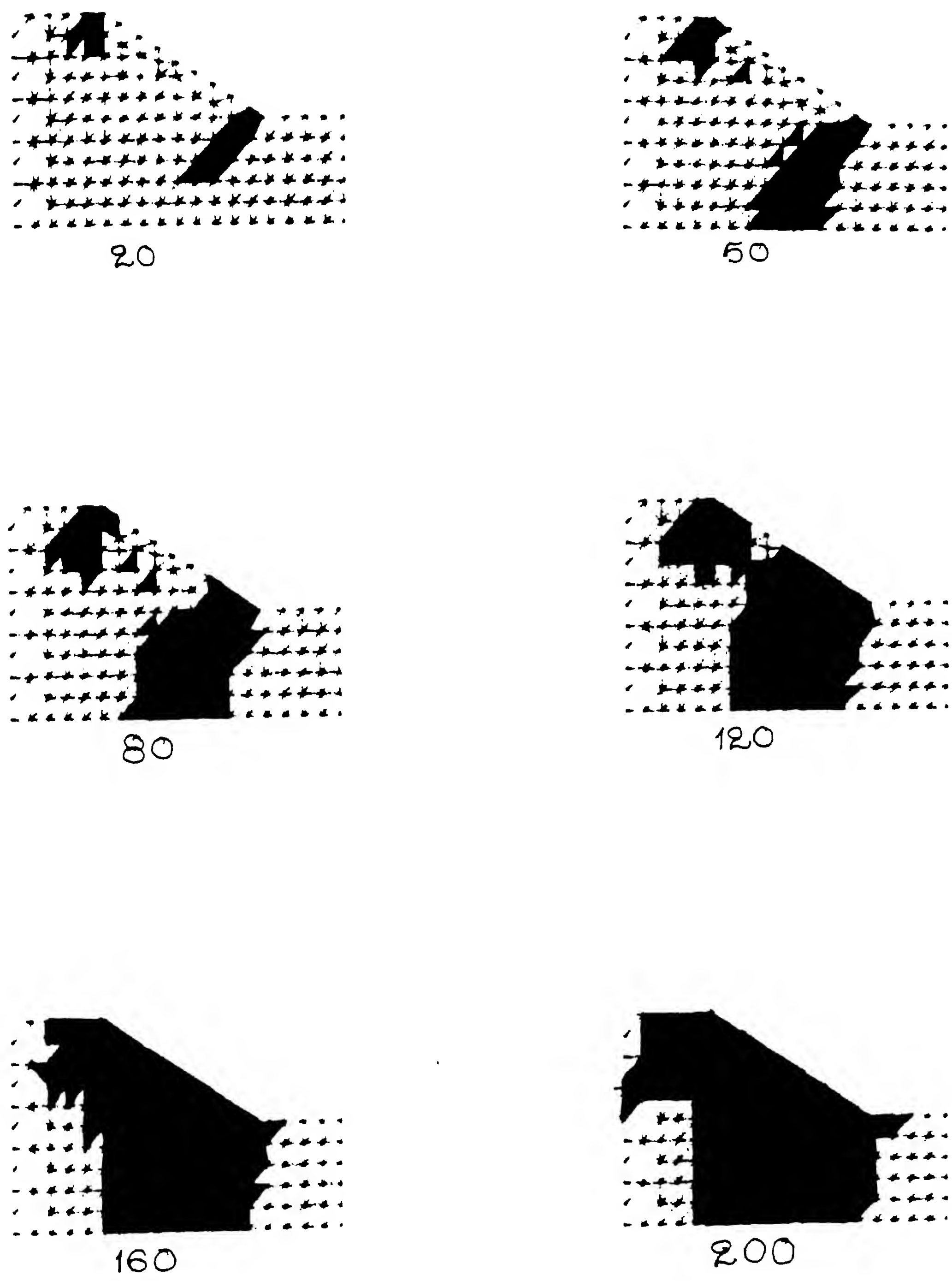


Fig. 5.14 Spreading of plastic zone for static axisymmetric case

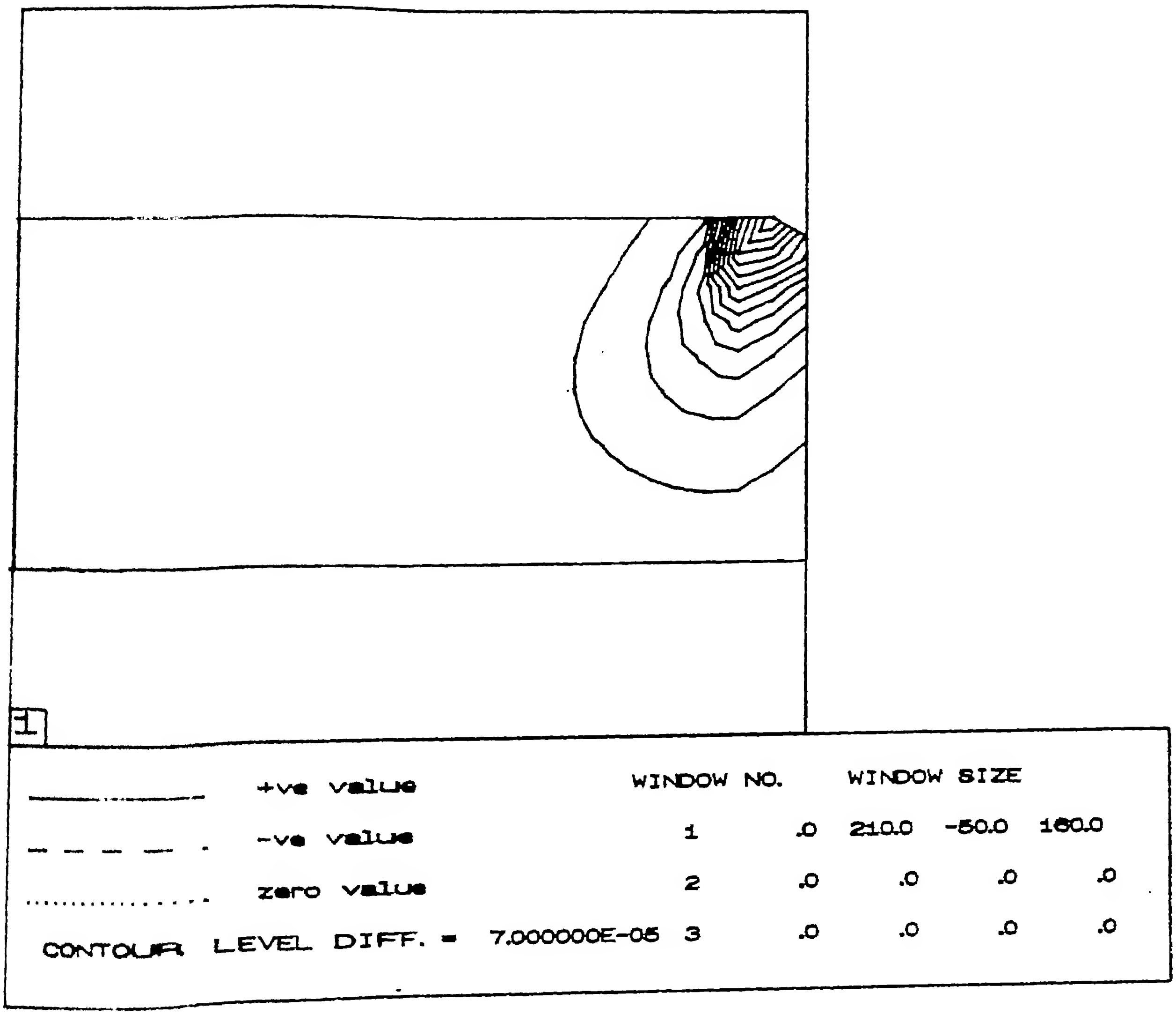


Fig. 5.16 Iso-equivalent strain contours for transient plane-strain case

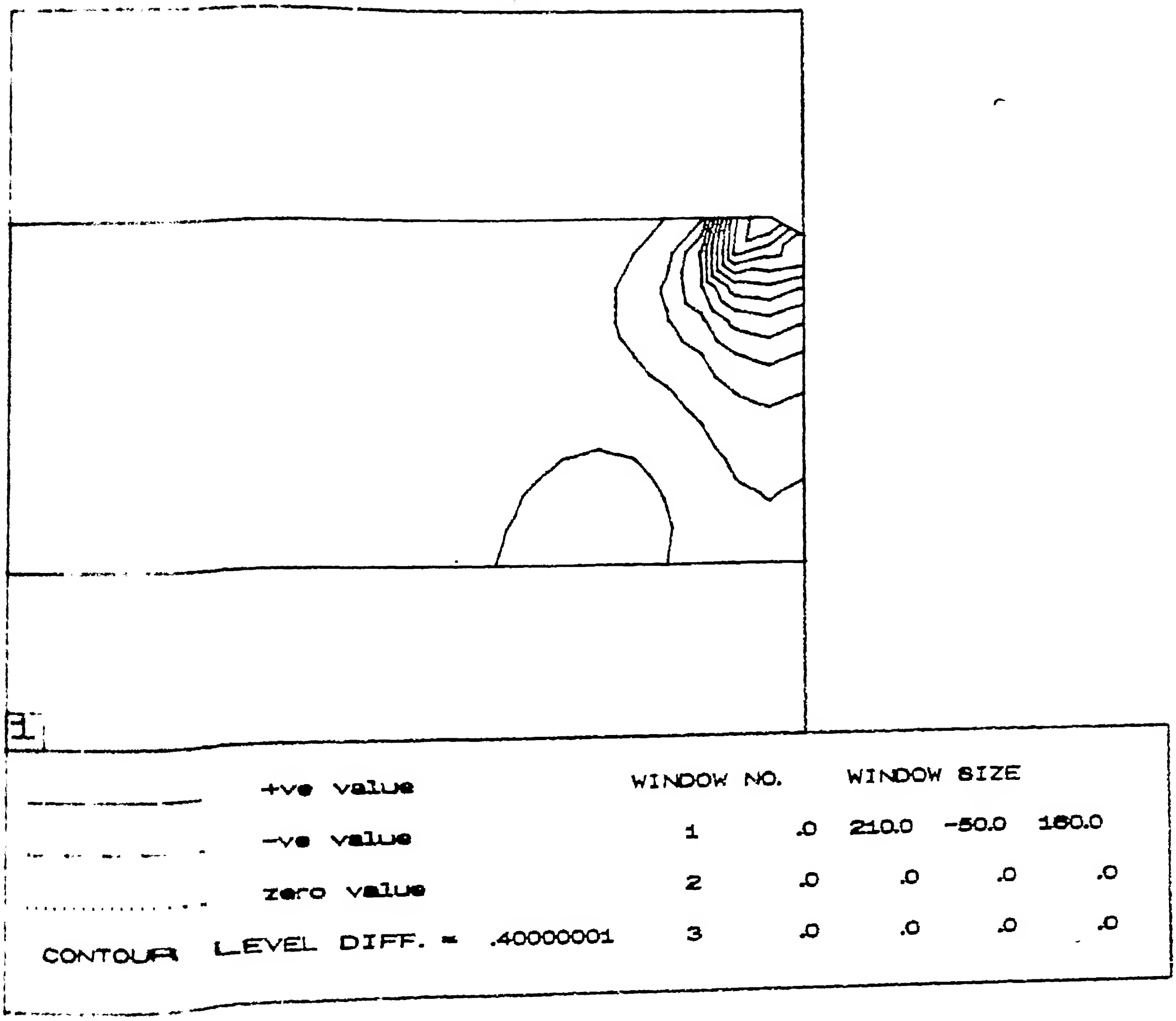


Fig. 5.17 Iso-equivalent stress contours for transient axisymmetric case

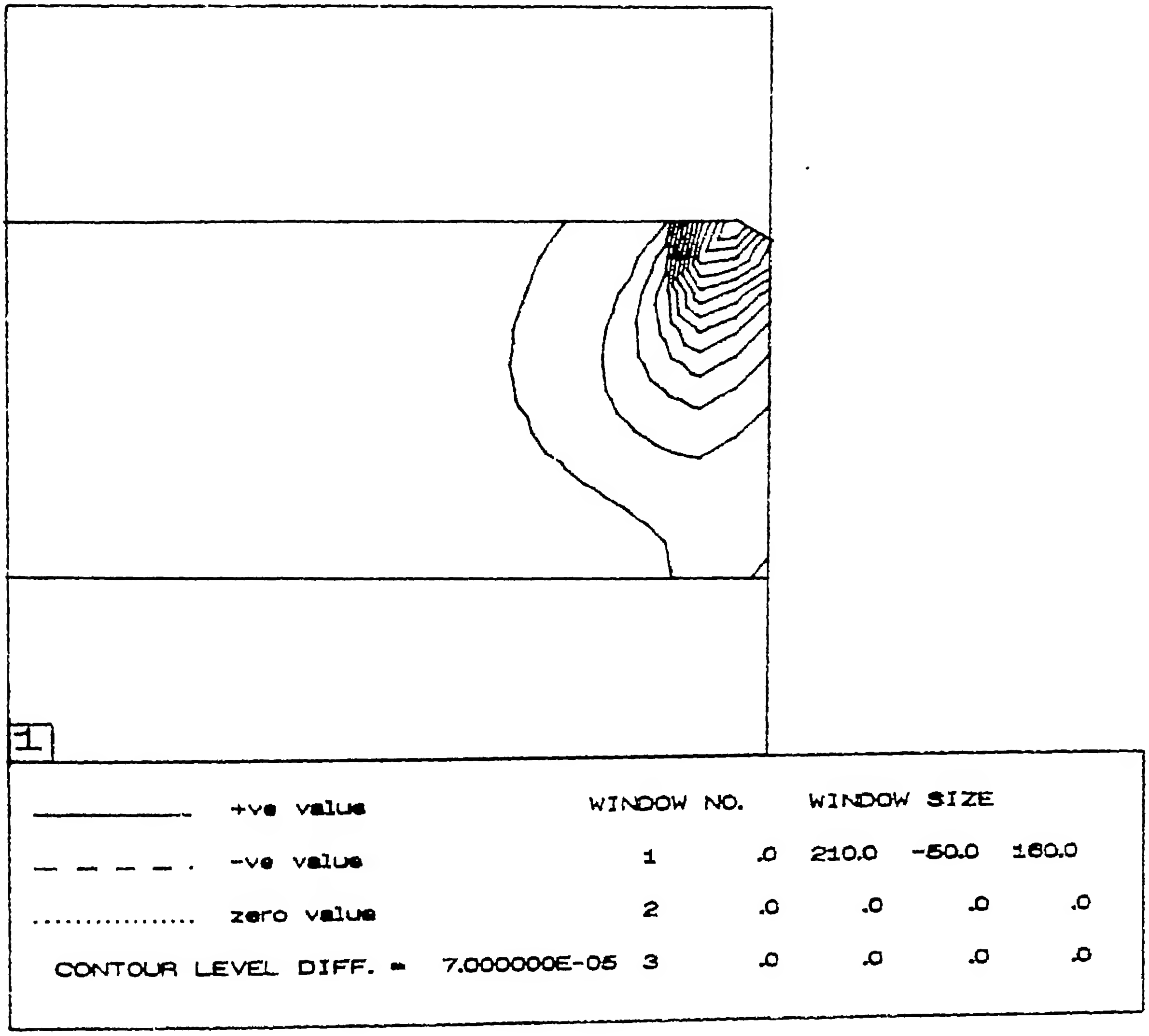


Fig. 5.18 Iso-equivalent strain contours for transient axisymmetric case

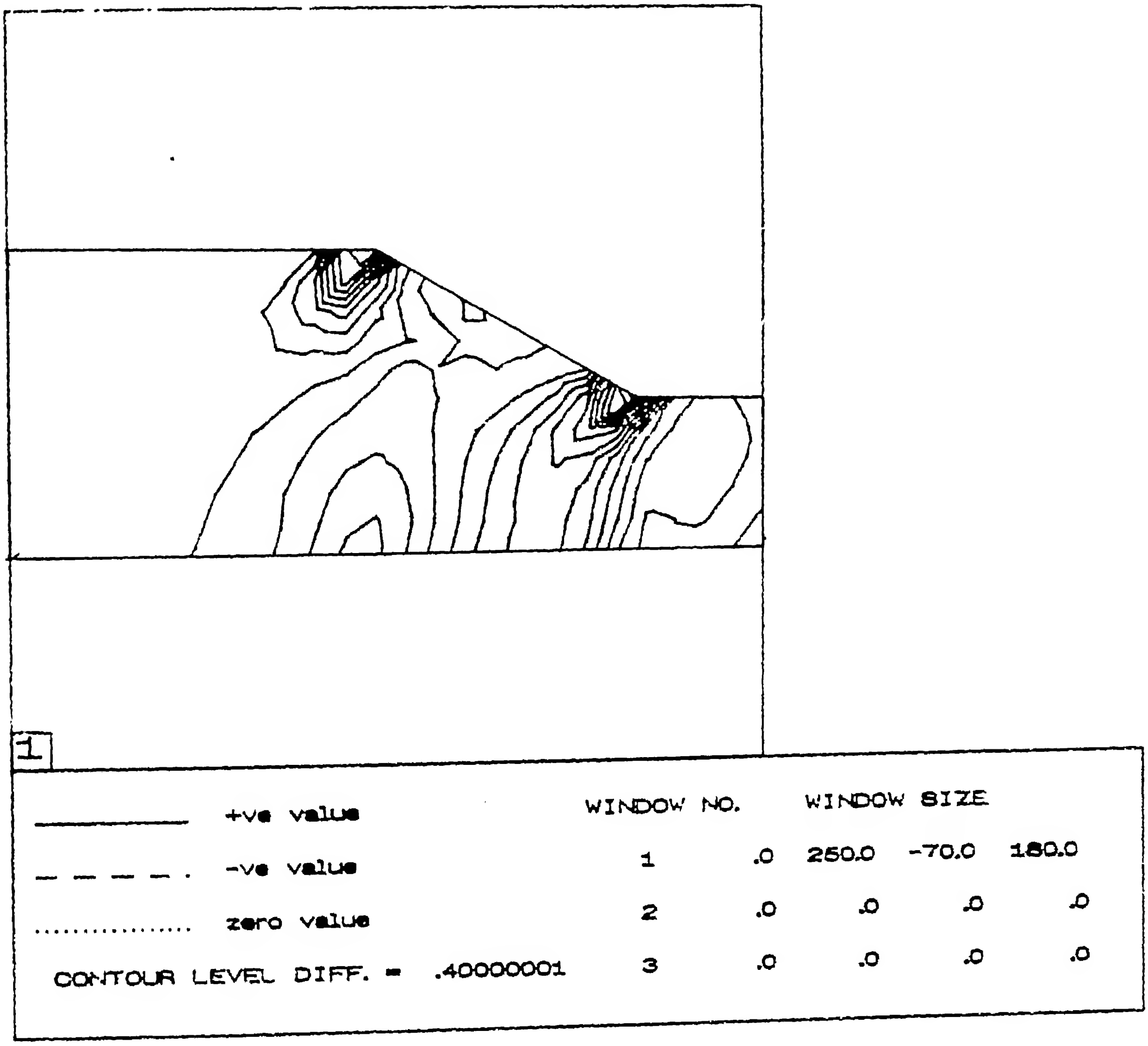


Fig. 5.19 Iso-equivalent stress contours for static plane-strain case

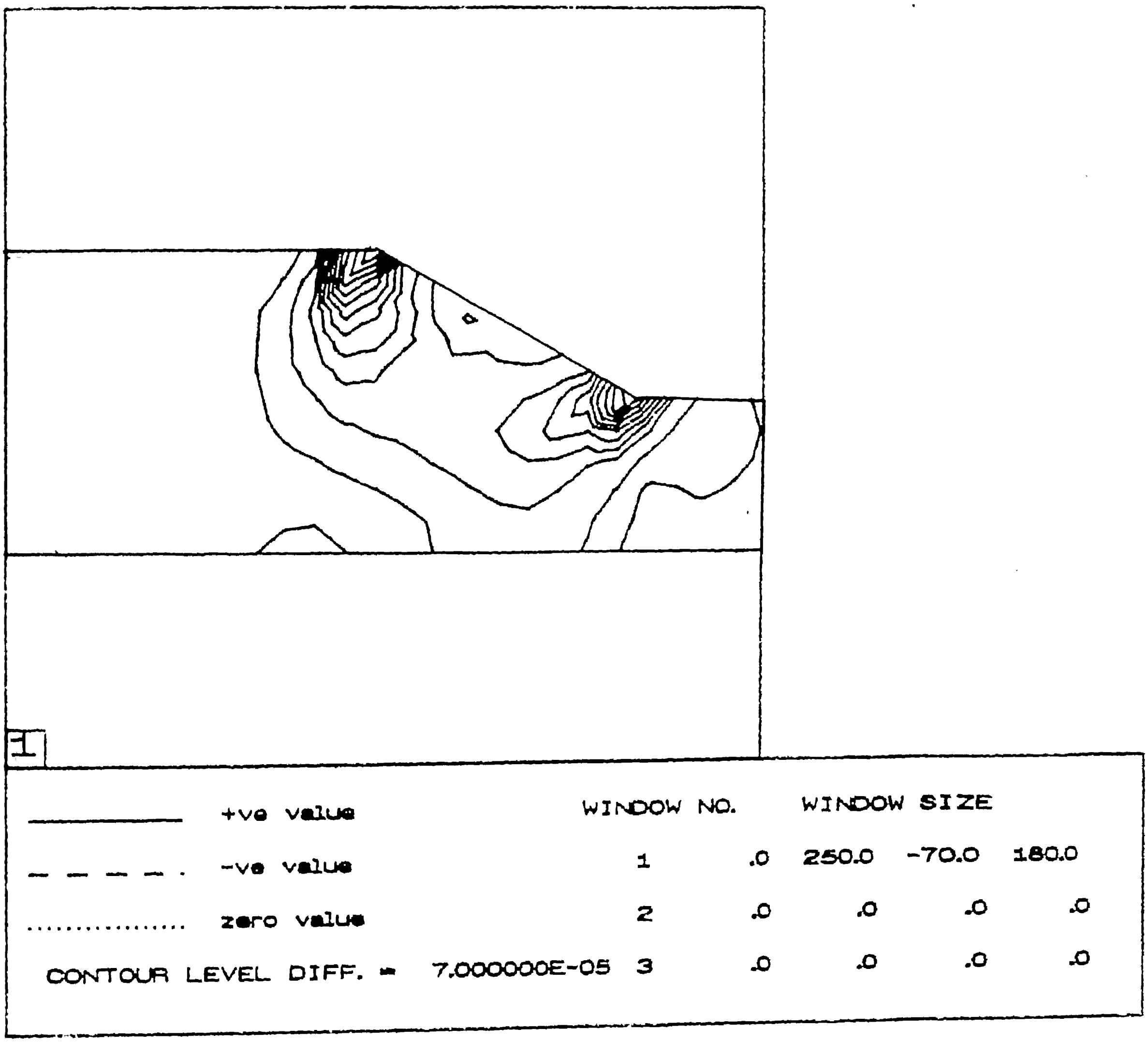


Fig. 5.20 Iso-equivalent strain contours for static plane-strain case

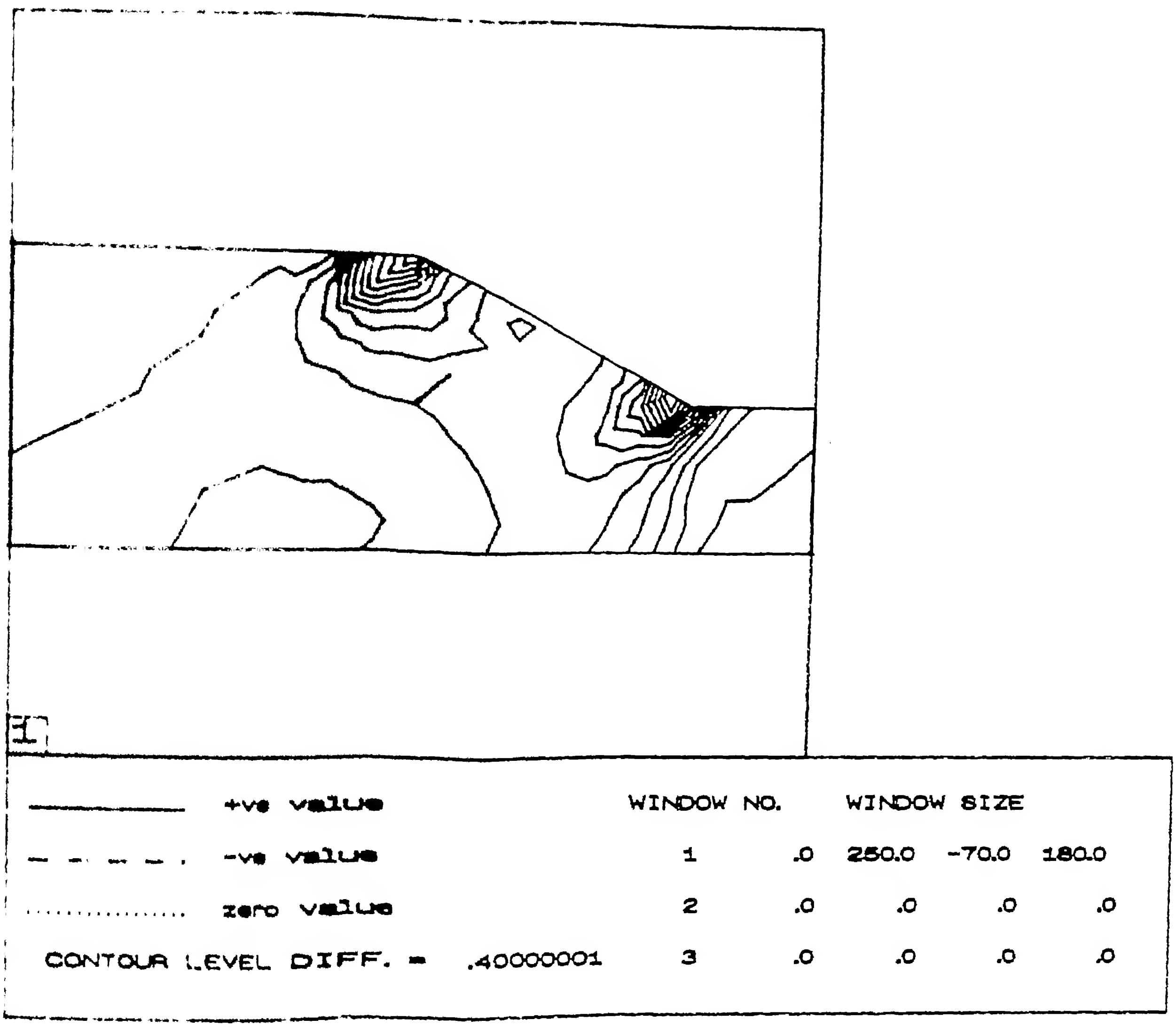


Fig. 5.21 Iso-equivalent stress contours for static axisymmetric case

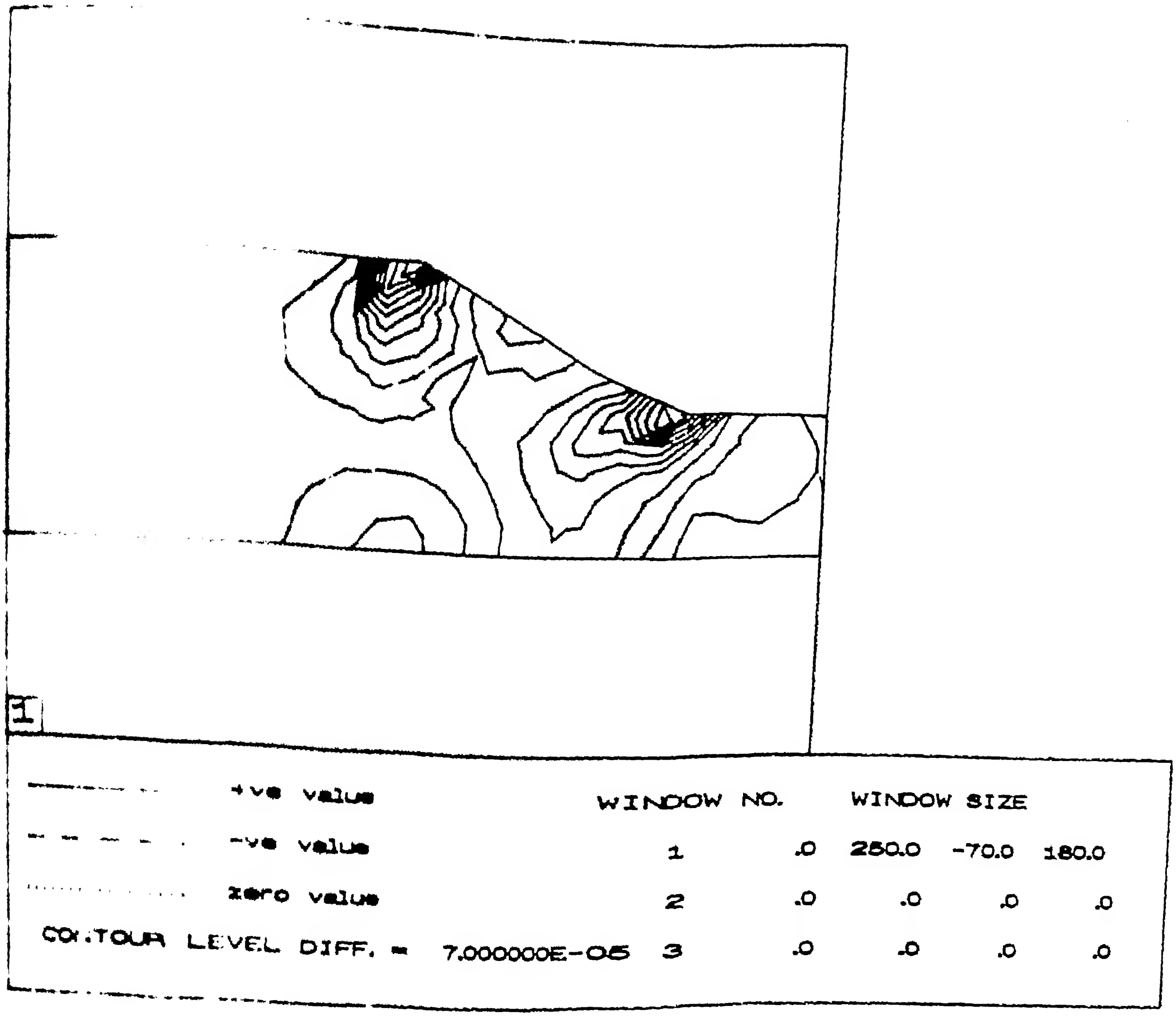


Fig. 5.22 Iso-equivalent strain contours for static axismmetric case

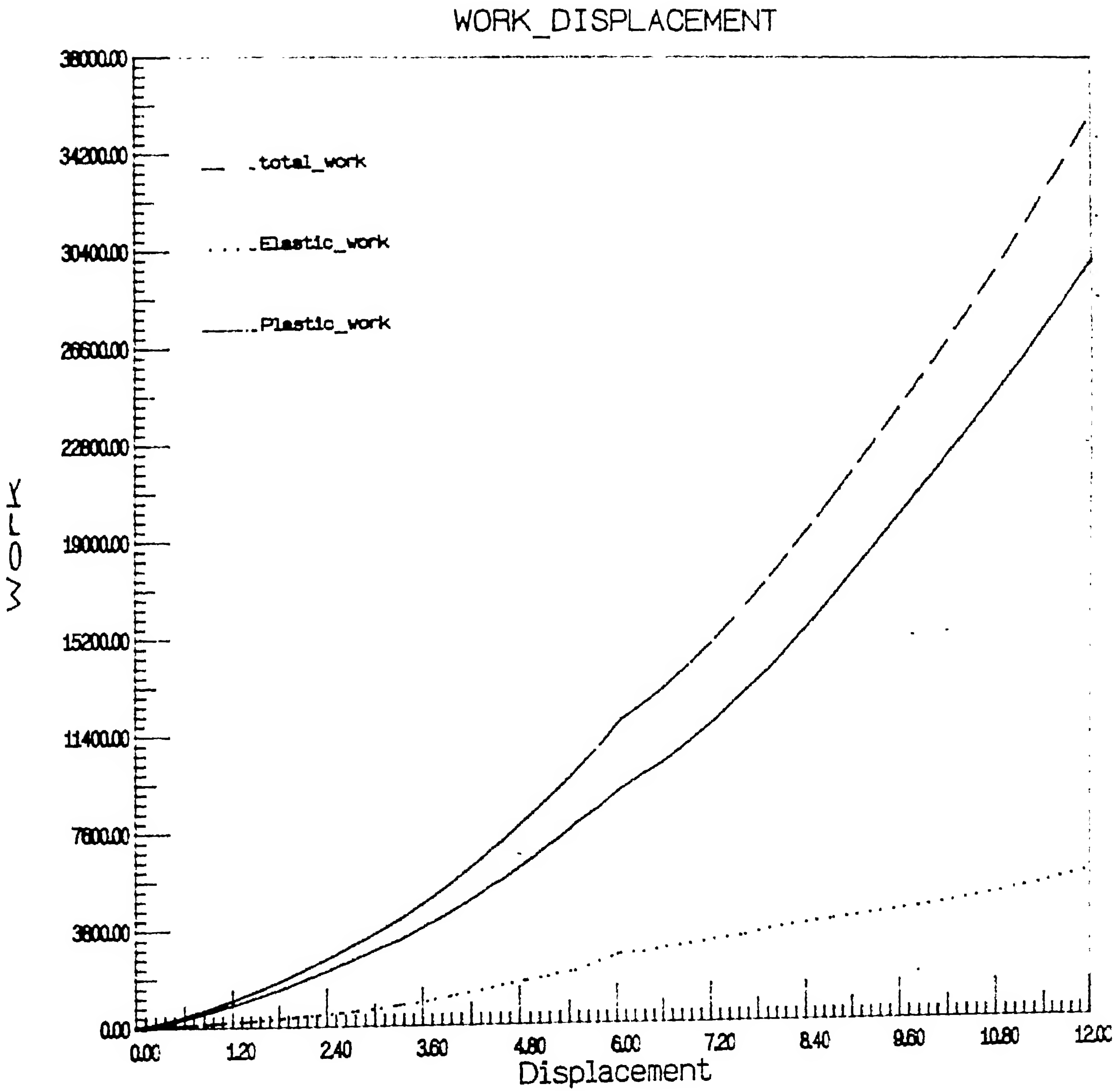


Fig. 5.23 Graph between the different work expenditure and the punch displacements

CHAPTER 6

CONCLUSIONS AND SCOPE FOR FUTURE WORK

CONCLUSIONS

The following conclusions can be drawn from the present study.

1. When the unloading is not considered at the exit of the extrusion die, it is observed that the entire specimen yields.

2. For conical die extrusion problem elastic work is comparable with plastic work.

3. For axisymmetric extrusion plastic zone is spreading more faster than the plane strain extrusion at the exit of the die.

SCOPE FOR FURTHER WORK

1. Higher order elements may be used to take care of high stress gradient.

2. Die geometry optimization.

3. Because of plastic deformations, the variation in temperature and resulting change in properties may be taken care of.

4. This Analysis may be extended to metal forming processes like forging, sheet metal forming etc.

5. The following areas of study may also be considered

— High stress concentration problems.

— Ductile fracture mechanics problems.

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APPENDIX -I

Derivation for $d\lambda$:

Consider a simple uniaxial tensile test,

$$\frac{d\varepsilon^{p}_{x}}{\sigma'_{x}} = d\lambda \qquad (1a)$$

Because

$$\Delta\bar{\varepsilon}^{\,p} = \sqrt{\frac{2}{3}\Delta\varepsilon_{ij}\Delta\varepsilon_{ij}}$$

$$= \sqrt{\frac{2}{3}\left[(\Delta\varepsilon^{p}_{1} - \Delta\varepsilon^{p}_{2})^{2} + (\Delta\varepsilon^{p}_{1} - \Delta\varepsilon^{p}_{2})^{2} + (\Delta\varepsilon^{p}_{1} - \Delta\varepsilon^{p}_{2})^{2}\right]}$$

Here $\Delta\varepsilon^{p}_{1} = -2\Delta\varepsilon^{p}_{2} = -2\Delta\varepsilon^{p}_{3}$

subscripts 1, 2 and 3 represents the principal directions.

$\therefore \quad \Delta\bar{\varepsilon}^{\,p} = \Delta\varepsilon^{p}_{1}$

Substituting back to equation (1a)

We get,

$$\frac{d\bar{\varepsilon}^{\,p}}{\sigma'_{x}} = d\lambda \qquad (1b)$$

$$d\bar{\varepsilon}^{\,p} = d\lambda\ \sigma'_{x}$$

$$= d\lambda\left(\sigma_{x} - \frac{\sigma_{x} + \sigma_{y} + \sigma_{z}}{3}\right)$$

$$= d\lambda \, \frac{2}{3} \, \sigma_x \qquad \text{since } \sigma_y = \sigma_z = 0$$

$$= d\lambda \, \frac{2}{3} \, \bar{\sigma}$$

or

$$d\lambda = \frac{3}{2} \, \frac{d\bar{\varepsilon}^{\,p}}{\bar{\sigma}} = \frac{3 \, d\bar{\sigma}}{2 \, \bar{\sigma} \, d\bar{\sigma}/d\bar{\varepsilon}^{\,p}}$$

$$d\lambda = \frac{3}{2} \, \frac{1}{\bar{\sigma}} \, \frac{d\bar{\sigma}}{H'} \tag{2}$$

APPENDIX - II

$$\text{Equivalent stress} \quad \bar{\sigma} = \sqrt{\frac{3}{2}\sigma'_{ij}\,\sigma'_{ij}} \qquad (1)$$

$$\bar{\sigma} + \Delta\bar{\sigma} = \sqrt{\frac{3}{2}(\sigma'_{ij} + \Delta\sigma'_{ij})\,(\sigma'_{ij} + \Delta\sigma'_{ij})}$$

Where

$\Delta\sigma$ is the increment of the stresses due to the increment of load.

$$\bar{\sigma} + r\Delta\bar{\sigma} = \sqrt{\frac{3}{2}\,(\sigma'_{ij} + r\Delta\sigma'_{ij})\,(\sigma'_{ij} + r\Delta\sigma'_{ij})}$$

$\Delta\sigma$ is scaled by the factor r so that the equivalent stress $\bar{\sigma} + r\Delta\bar{\sigma}$ is just to reach yield stress(Y)

$$\therefore \quad Y = \bar{\sigma} + r\Delta\bar{\sigma}$$

$$Y^2 = \frac{3}{2}\left[\sigma'_{ij}\,\sigma'_{ij} + 2\,r\,\sigma'_{ij}\,\Delta\sigma'_{ij} + r^2\,\Delta\sigma'^{\,2}_{ij}\right]$$

$$\text{Let} \quad \Delta\bar{\sigma}^2_{step} = \frac{3}{2}\,\Delta\sigma'_{ij}\,\Delta\sigma'_{ij}$$

$$Y^2 = \bar{\sigma}^2 + 2\,r\left[(\bar{\sigma} + \Delta\bar{\sigma})^2 - \bar{\sigma}^2 - \Delta\bar{\sigma}^2_{step}\right] + r^2\,\Delta\bar{\sigma}^2_{step}$$

Substituting for $\bar{\sigma} + \Delta\bar{\sigma}$ from equation (2), the above equation becomes

$$Y^2 - \bar{\sigma}^2 - 2\,r\,\Gamma + r^2\,\Delta\bar{\sigma}^2_{step} = 0 \qquad (3)$$

Where

$$\Gamma = \Delta\bar{\sigma}_{step}^{2} - 2\,\bar{\sigma}\,\Delta\bar{\sigma} - \Delta\bar{\sigma}^{2}$$

Equation (3) is quadratic in r. By solving equation (3)

We get

$$r = \frac{\Gamma + \sqrt{\Gamma^2 - (Y^2 - \bar{\sigma}^2)\,\Delta\bar{\sigma}_{step}^{2}}}{\Delta\bar{\sigma}_{step}^{2}} \qquad (4)$$